Journal of Statistical Physics, Vol. 105, Nos. 1/2, October 2001 (© 2001)

Adsorbing and Collapsing Directed Animals

E. J. Janse van Rensburg[1] **and A. Rechnitzer**[1]

Received April 17, 2001

A model of a self-interacting directed animal, which also interacts with a solid wall, is studied as a model of a directed branched polymer which can undergo both a collapse and an adsorption transition. The directed animal is confined to a 45° wedge, and it interacts with one of the walls of this wedge. The existence of a thermodynamic limit in this model shown, and the presence of an adsorption transition is demonstrated by using constructive techniques. By comparing this model to a process of directed percolation, we show that there is also a collapse or θ-transition in this model. We examine directed percolation in a wedge to show that there is a collapse phase present for arbitrary large values of the adsorption activity. The generating function of adsorbing directed animals in a half-space is found next from which we find the tricritical exponents associated with the adsorption transition. A full solution for a collapsing directed animal seems intractible, so instead we examine the collapse transition of a model of column convex directed animals with a contact activity next.

KEY WORDS: Directed animals; directed percolation; branched polymer adsorption; generating functions; convex directed animals.

1. INTRODUCTION

The introduction of lattice animals (connected subgraphs of a lattice) dates back to the combinatorial problem of cell growth.(38, 47) These objects also arise in studies of percolation,(55) and in the physical sciences are often used as models of branched polymers in dilute solution.(48) Self-interacting models of lattice animals have received particular attention over the last decades as models of branched polymers undergoing a θ-collapse,(16, 27, 28, 31, 32) or an adsorption transition on a plane or at an interface between emiscible solvents (defect plane). There is a connection between the phenomenon of percolation and models of collapsing lattice

---

[1] Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada.

animals,(19, 27, 28, 33, 42) and this has been exploited to show that the limiting free energy in certain models of collapsing lattice animals is a non-analytic function, which implies that the model undergoes a collapse transition.

The adsorption problem in models of polymers was introduced by Hammersley *et al.*,(36) see also the papers by Duplantier and Saleur,(20) Cardy,(13) and Batchelor and Yung.(1) A lattice tree model of adsorbing branched polymers was considered in Janse van Rensburg and You;(45) see also De'Bell and Lookman.(15) In all these models it can be demonstrated that there is a non-analyticity in the limiting free energy corresponding to the adsorption transition, and moreover, for adsorption onto a surface from a half-space, the transition takes place at a nontrivial value of the adsorption activity(40, 43, 45) (for similar results for adsorbing copolymers, see You and Janse van Rensburg.(57)) More on this can be found in the work of Eisenreigler and collaborators,(21–23) and see also Fisher.(25)

Directed models of adsorbing and collapsing polymers have also been considered. Directed versions of linear polymers were introduced by Privman and Švrakić,(52, 53) and many variations of this model have been studied. A model of adsorbing directed walks was studied by Carvalho and Privman,(14) Privman *et al.*,(51) and a collapsing model of a directed walk was solved by Brak *et al.*;(10, 11) the full solution shows that this model is very similar to models of inflating convex and partially convex lattice vesicles (these are models of square lattice polygons with area and perimeter activities; increasing the area activity takes the vesicle through a transition into an inflated phase): see for example Pólya,(49) Bousquet-Mélou,(3, 4) Brak *et al.*,(12) Prellberg and Brak.(50) The adsorption of a fully directed walk was reviewed by Janse van Rensburg,(41) and a model of adsorbing and collapsing columns was found to be very similar to both models of inflating partially convex vesicles and adsorbing (partially) directed walks(44) (see also Brak *et al.*(9)).

In this paper we consider a model of interacting directed animals in the square lattice. The directed animal is confined to a wedge, and it interacts with the main diagonal through an adsorption activity which is conjugate to the number of visits the animals makes to the main diagonal. The model becomes more interesting with the introduction of cycle and contact activities; and one would expect that increasing these activities will take the directed animal through a θ-transition into a collapsed phase. In Section 2 we consider the adsorption transition in this model. We show the existence of a limiting free energy, and prove that there is a collapse transition in the model, regardless of the strengths of the cycle and contact activities. In Section 3 we focus instead on the θ-transition in this model. We show that it is closely related to a percolation phenomenon in a certain model of site-bond directed percolation. We show that this model of directed percolation implies that the limiting free energy is a non-analytic function of the cycle and/or contact activities, and that a phase transition, which presumably is a θ-transition, occurs at all values of the adsorption activity which correspond to a repulsive main diagonal. We then demonstrate that collapse occurs when the main diagonal is attractive, and find the presence of a collapsed phase for arbitrarily large values of the adsorption activity.

It might be tempting to suppose that there should be a simple generating function for this model (since it is fully directed), but unfortunately, this seems not to be the case. In the absence of cycle and contact activites one can find a generating function with an adsorption activity, from which we may calculate critical exponents associated with the adsorption activity (see Section 4.1). However, the introduction of a cycle and/or a contact activity changes the picture dramatically and we are unable to find a solution. In light of this we consider a simpler model of collapsing directed column-convex animals and find a generating function in terms of q-deformed Bessel functions; the analysis of such functions is quite difficult and even in the case of this simplified model we are not able to completely determine its phase diagram. This can be seen to imply that directed animals, while seemingly a simple approximation to lattice animal models of branched polymers, are in their own right a complicated model which is ordinarily unlikely to have a generating function which is algebraic (and therefore easily solvable).

2. DIRECTED ANIMALS IN A HALF-SPACE

The square lattice can be directed by assigning to each horizontal edge the direction "East" and to each vertical edge the direction "North." Every directed edge in the directed lattice has an initial vertex, and a final vertex (and a directed edge is "from" its initial vertex "to" its final vertex). A vertex is a *source* in any subgraph of the directed square lattice if there are no directed edges (in the subgraph) to it, but there is at least one directed edge from it (we say that its *indegree* is zero, and its *outdegree* is bigger than zero). Every vertex has two *descendents* which are the endpoints of the directed edges pointing from it. The initial vertices of the two directed edges pointing to a vertex are called its *predecessors*.

A connected section graph of the directed square lattice is a directed animal. Ordinarily, we shall only consider directed animals with one source vertex which is located at the origin. Notice that there is a path of directed edges from the origin to any vertex in such a directed animal. The animal is a *site animal* in the sense that two adjacent vertices (or *sites*) are within the animal if and only if the edge joining them is as well.

A *positive* directed animal is a directed animal such that all its vertices lie above or on the main diagonal in the square lattice (the subset of vertices in the square lattice confined to the first quadrant and above or on the main diagonal will be called the *principle wedge*; all vertices in this wedge have non-negative coordinates (x, y) with $y \geqslant x$). The positive directed animals in this paper will normally be confined to the principle wedge, and will interact with the main diagonal. One such animal is illustrated in Fig. 1. Observe that the lattice and the animal is oriented such that the main diagonal is horizontal. The source of the animal is at the origin, and the animal is otherwise confined to the half-space above the main diagonal, which we shall call the *positive half-space*.

The directed animal in Fig. 1 has at least one vertex (its source) within a distance of one lattice spacing from the main diagonal. Such an animal will be called *attached*. If the directed animal in Fig. 1 is translated away from the main diagonal, then it will eventually not be attached anymore, and its source will not be located at the origin. In some constructions we shall encounter attached animals, but whose source vertex is a certain height from the main diagonal.

The vertices in a directed animal can be lexicographically ordered, first in the direction along the main diagonal, and then perpendicular to it. Such an ordering will produce a first and last vertex. The first vertex will always be the source, and is also called the *bottom vertex*. The last vertex is the *top vertex*.

A *contact* in a directed animal will be a pair of vertices with coordinates of the form (x, y) and $(x+1, y-1)$; some contacts are indicated in Fig. 1 as dotted lines. A *visit* in a positive and directed animal is a vertex which lies in the main diagonal, but we shall not count the source vertex as a visit, even if it is located at the origin (and thus in the main diagonal). The number of *cycles* in the directed animal is equal to the number of vertices in the animal with two occupied predecessors.

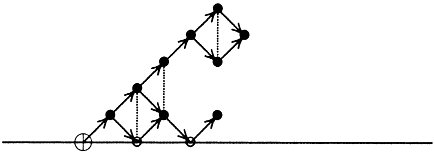

Fig. 1. An attached, positive and directed animal with 12 vertices, 2 visits, 2 cycles and 13 edges. The source vertex is at the origin in this animal, but we do not count it as a visit. Notice that the usual Cartesian axes have been rotated throught 45° so that the main diagonal is horizontally oriented.

Let $a_v(c, k, w)$ be the number of positive and directed animals with one source at the origin and with v vertices, c cycles, k contacts and w visits. The quantities v, c and w satisfy the relations

$$\begin{aligned} n+1-v &= c \\ n+w+q &= 2v-1 \end{aligned} \tag{2.1}$$

where n is the number of edges in the underlying graph of the animal, and q is the number of perimeter edges *in the positive halfspace* leaving from a site within the animal and arriving at vertices outside it.[2] The first equation is Euler's relation, while the second equation follows since every vertex that is not a visit in the animal is a source of two edges, each one either an edge in the animal, or an edge in the perimeter. Each visit is the source of a directed edge above the main diagonal, and this edge is either in the animal, or in the perimeter of the animal, or is a directed edge which is below the diagonal which is neither in the animal nor in its perimeter.

We define the partition function of an interacting model of a positive and directed animal adsorbing in the main diagonal as:

$$Z_v(x, y, z) = \sum_{n, k, w \geqslant 0} a_v(c, k, w)\, x^c y^k z^w. \tag{2.2}$$

In the next section the existence of a thermodynamic limit in this model is proven.

2.1. Existence of the Thermodynamic Limit

The existence of a thermodynamic limit in models of animals usually relies on a supermultiplicative relation for the partition function which is obtained by concatenation(34, 35, 47) and a most popular height argument.(36, 37) For directed animals, however, the proof follows somewhat similar lines, but is greatly complicated by the absence of reflection symmetries; as it will become apparent below.

In this section it will be necessary to relax the condition that the source of the directed animals be located at the origin. We define the *height* of a vertex to be the distance between the vertex and the main diagonal, in units of $1/\sqrt{2}$ lattice spacings (so that it is always an integer). We are only intested in the heights of the bottom vertex (source) and the top vertex in any given directed animal (see Fig. 2).

---

[2] Each visit in the main diagonal is incident with a directed edge below the diagonal. These edges are *not* considered part of the perimeter. In other words, the perimeter edges are all those edges incident with the animal which could become part of the animal.

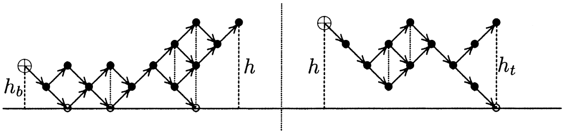

Fig. 2. Two attached, positive and directed animals can be concatenated if the top vertex of the first has the same height as the bottom vertex (source) of the second. The construction proceeds by first translating the animal on the right until its source is $\sqrt{2}$ lattice units to the right of the top vertex of the animal on the left. One vertex can be added between the two animals to join them into a single animal. No new contacts or cycles are created in this construction, and the number of visits is just the sum of the visits of the two animals. The total number of vertices also increases by one, and only a subclass of attached, positive and directed animals can be created in this way. Contacts in these directed animals are indicated by dotted lines, and visits by open circles in the main diagonal. The sources are indicated by $\oplus$.

Suppose that the height of the bottom vertex is h_b, and the height of the top vertex is h_t, in a certain animal. Let $a_v(c, k, w; [h_b h_t])$ be the number of directed animals with source (bottom vertex) of height h_b and top vertex of height h_t. Then $a_v(c, k, w) = \sum_h a_v(c, k, w; [0h])$. We show in Fig. 2 that two attached, positive and directed animals can be concatenated into a single attached, positive and directed animal if the height of the top vertex of the first is the same as the height of the source (bottom vertex) of the second. The construction is described in the caption of Fig. 2. If the first animal has v_1 vertices, c_1 cycles, k_1 contacts and w_1 visits, and the second animal has v_2 vertices, $c-c_1$ cycles, $k-k_1$ contacts and $w-w_1$ visits, then the resulting animal has v_1+v_2+1 vertices, c cycles, k contacts, and w visits. This gives the following supermultiplicative relation

$$\sum_{c_1, k_1, w_1} a_{v_1}(c_1, k_1, w_1; [h_b h])\, a_{v_2}(c-c_1, k-k_1, w-w_1; [h h_t]) \leqslant a_{v_1+v_2+1}(c, k, w; [h_b h_t]). \tag{2.3}$$

Multiplying by $x^c y^k z^w$, and summing over c, k and w, gives

$$Z_{v_1}(x, y, z; [h_b h])\, Z_{v_2}(x, y, z; [h h_t]) \leqslant Z_{v_1+v_2+1}(x, y, z; [h_b h_t]). \tag{2.4}$$

If we now put $h_b = h_t = h$, then the following limit exists[39]

$$\mathscr{F}_{hh}(x, y, z) = \lim_{v \to \infty} \frac{1}{v} \log Z_v(x, y, z; [hh]), \tag{2.5}$$

since $-\log[Z_{v-1}(x, y, z; [hh])]$ satisfies a standard subadditivity relation. Observe that $Z_v(x, y, z; [hh]) \leqslant \lambda^v \max\{1, (xyz)^v\}$, where λ is the growth constant of lattice animals in the square lattice, so that $\mathscr{F}_{hh}(x, y, z)$ is finite for finite values of its arguments.

On the other hand, we can now show that $\mathscr{F}_{00}(x, y, z)$ is in fact the limiting free energy of the original model, which has partition function $\sum_h Z_v(x, y, z, [0h])$. Define

$$Z_v(x, y, z) = \sum_h Z_v(x, y, z; [0h]). \tag{2.6}$$

There is a most popular value for h in the right hand side of Eq. (2.6). Let this be h_t^* (and notice that h_t^* is the most popular value of the top vertex height, and may depend on v, and on the activities x, y and z). Then, since h takes on at most v values, we have

$$Z_v(x, y, z; [0h_t^*]) \leqslant Z_v(x, y, z) \leqslant v Z_v(x, y, z; [0h_t^*]). \tag{2.7}$$

On the other hand, we may concatenate $M+2$ animals together as in Fig. 2, with the first tree counted by $Z_v(x, y, z; [0h_t^*])$, the next M trees counted by $Z_v(x, y, z; [h_t^* h_t^*])$, and the last tree counted by $Z_v(x, y, z; [h_t^* 0])$. This gives the following inequality:

$$\begin{aligned} &Z_v(x, y, z; [0h_t^*])[Z_v(x, y, z; [h_t^* h_t^*])]^M Z_v(x, y, z; [h_t^* 0]) \\ &\quad \leqslant Z_{M(v+1)+1}(x, y, z; [00]). \end{aligned} \tag{2.8}$$

Take the logarithm of this, divide by $M(v+1)+1$, and take $M \to \infty$. By Eq. (2.5) we obtain

$$\frac{1}{v+1} \log Z_v(x, y, z; [h_t^* h_t^*]) \leqslant \mathscr{F}_{00}(x, y, z). \tag{2.9}$$

We can now safely take $v \to \infty$ on the left hand side in Eq. (2.9) to see that

$$\mathscr{F}_{h_t^* h_t^*}(x, y, z) \leqslant \mathscr{F}_{00}(x, y, z) \leqslant \liminf_{v \to \infty} \frac{1}{v} \log Z_v(x, y, z; [0h_t^*]), \tag{2.10}$$

since h_t^* is the most popular value of h in Eq. (2.6).

The following lemma will be critical in establising the existence of the thermodynamic limit. It relates $Z_v(x, y, z; [0h_t^*])$ and $Z_v(x, y, z; [h_t^* 0])$ to one another, where h_t^* is that most popular value of h in $Z_v(x, y, z; [0h])$.[3]

Lemma 2.1. It is the case that

$$\limsup_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [0h_t^*]) = \limsup_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [h_t^* 0]).$$

Proof. To see this, we use an argument similar to that leading to Eq. (2.8). Concatenate, in an alternating fashion, animals counted by $Z_v(x, y, z; [0h_t^*])$ and $Z_v(x, y, z; [h_t^* 0])$. By Eq. (2.4), this shows that

$$\begin{aligned} &[Z_v(x, y, z; [0h_t^*])]^M \, [Z_v(x, y, z; [h_t^* 0])]^{M-1} \\ &\qquad \leqslant Z_{(2M-1)(v+1)-1}(x, y, z; [0h_t^*]); \\ &[Z_v(x, y, z; [h_t^* 0])]^M \, [Z_v(x, y, z; [0h_t^*])]^{M-1} \\ &\qquad \leqslant Z_{(2M-1)(v+1)-1}(x, y, z; [h_t^* 0]). \end{aligned}$$

Notice that h_t^* is that value of h which is the most popular value in $Z_v(x, y, z; [0h])$. Take logarithms, divide by $(2M-1)(v+1)-1$, and take the lim inf as $M \to \infty$ on the right hand sides, while h_t^* is kept fixed (since v is fixed). This shows that

$$\begin{aligned} &\frac{1}{2v+2} \log Z_v(x, y, z; [0h_t^*]) + \frac{1}{2v+2} \log Z_v(x, y, z; [h_t^* 0] \\ &\qquad \leqslant \liminf_{M\to\infty} \frac{1}{(2M-1)(v+1)-1} \log Z_{(2M-1)(v+1)-1}(x, y, z; [0h_t^*]); \\ &\frac{1}{2v+2} \log Z_v(x, y, z; [h_t^* 0]) + \frac{1}{2v+2} \log Z_v(x, y, z; [0h_t^*] \\ &\qquad \leqslant \liminf_{M\to\infty} \frac{1}{(2M-1)(v+1)-1} \log Z_{(2M-1)(v+1)-1}(x, y, z; [h_t^* 0]). \end{aligned}$$

Naturally, it is also the case that

$$\begin{aligned} &\liminf_{M\to\infty} \frac{1}{(2M-1)(v+1)-1} \log Z_{(2M-1)(v+1)-1}(x, y, z; [0h_t^*]) \\ &\qquad \leqslant \limsup_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [0h_t^*]), \end{aligned}$$

---

[3] In the case of an undirected model, this result would be immediate by reflection symmetry, but the directed nature of the model here makes this non-trivial, since a directed animal reflected throught a line perpendicular to the prefered direction is not necessarily a directed animal.

where h_t^* is now the most popular value of h in $Z_v(x, y, z; [0h])$ on the right hand side as well. Thus, if the $\limsup_{v\to\infty}$ is taken on the left-hand sides of the penultimate equations, then

$$\limsup_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [h_t^* 0]) \leqslant \limsup_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [0h_t^*]);$$

$$\limsup_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [0h_t^*]) \leqslant \limsup_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [h_t^* 0]);$$

and this completes the lemma. ▮

Next, choose $h = 0$ and $h_b = h_t = h_t^*$ in Eq. (2.4) to see that

$$Z_v(x, y, z; [h_t^* 0])\, Z_v(x, y, z; [0h_t^*]) \leqslant Z_{2v+1}(x, y, z; [h_t^* h_t^*]). \tag{2.11}$$

Taking logarithms, dividing by $2v$ and letting $v \to \infty$, while using Lemma 2.1, gives eventually

$$\limsup_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [0h_t^*]) \leqslant \mathscr{F}_{h_t^* h_t^*}(x, y, z). \tag{2.12}$$

Comparison to Eq. (2.10) shows the existence of the desired limit, and we obtain

$$\lim_{v\to\infty} \frac{1}{v} \log Z_v(x, y, z; [0h_t^*]) = \mathscr{F}_{00}(x, y, z) \equiv \mathscr{F}(x, y, z). \tag{2.13}$$

A standard application of the Cauchy–Schwartz inequality shows that $\mathscr{F}(x, y, z)$ is also a convex function of anyone of $\log x$, or $\log y$, or $\log z$. Thus, $\mathscr{F}(x, y, z)$ is continuous, and differentiable almost everywhere.

2.2. Adsorption of the Directed Animal

It will be shown that $\mathscr{F}(x, y, z)$ is a non-analytic function. In particular, for any fixed values of x and y there is a critical value of z corresponding to an adsorption transition. To see this, first observe that if any animal is translated one step from the main diagonal in the vertical direction, then all visits are destroyed (see Fig. 3). The animal can be reconnected with the origin by adding a single vertex at the origin. The result is that

$$a_v(c, k, w) \leqslant a_{v+1}(c, k, 0). \tag{2.14}$$

The consequence of Eq. (2.14) is Lemma 2.2.

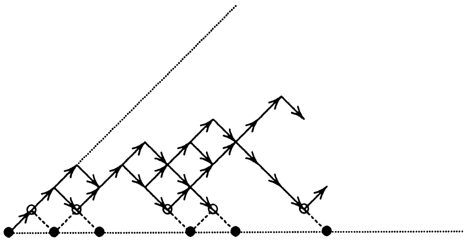

Fig. 3. Translating a positive and directed animal one step vertical will break all visits, but every old visit can be used to generate a new visit by adding edges along the dashed lines. This may create one new cycle and/or a new contact for every new visit.

Lemma 2.2. The partition function $Z_v(x, y, z)$ satisfies the following two inequalities:

$$Z_v(x, y, z) \leqslant \lceil v/2 \rceil [\max\{1, z\}]^{\lceil v/2 \rceil} Z_{v+1}(x, y, 0);$$

$$Z_v(x, y, z) \geqslant Z_v(x, y, 0).$$

Proof. The partition function of animals with no visits is given by $Z_v(x, y, 0)$. Since this is also the coefficient of z^0 in $Z_v(x, y, z)$, the second inequality is obtained. To find the first inequality, use Eq. (2.14). Since the number of visits is at most $\lfloor (v-1)/2 \rfloor = \lceil v/2 \rceil - 1$ in a positive and directed animal with v vertices, it follows that

$$Z_v(x, y, z) \leqslant \left[\sum_{c, k \geqslant 0} a_{v+1}(c, k, 0)\, x^c y^k \right] \left[\sum_{w \geqslant 0}^{\lceil v/2 \rceil - 1} [\max\{1, z\}]^w \right],$$

and this gives the first inequality above. ∎

Theorem 2.3. $\mathscr{F}(x, y, z)$ is a non-analytic function of z for each finite and fixed value of (x, y). In particular, $\mathscr{F}(x, y, z) = \mathscr{F}(x, y, 1)$ for all $0 < z \leqslant 1$, and there exists a $z_c(x, y) \geqslant 1$ such that $\mathscr{F}(x, y, z) \neq \mathscr{F}(x, y, 1)$ for $z > z_c(x, y)$.

Proof. If $z > 1$, then $Z_v(x, y, z) \geqslant a_v(0, 0, \lceil v/2 \rceil)\, z^{\lceil v/2 \rceil} \geqslant z^{\lceil v/2 \rceil}$, and so

$$\mathscr{F}(x, y, z) \geqslant \tfrac{1}{2} \log z, \qquad \text{if} \quad z \geqslant 1.$$

However, if $z \leqslant 1$, then the squeeze theorem for limits and Lemma 2.2 shows that

$$\mathscr{F}(x, y, z) = \lim_{v \to \infty} \frac{1}{v} \log Z_v(x, y, 0) = \mathscr{F}(x, y, 0), \qquad \text{if} \quad z \leqslant 1,$$

and in particular $\mathscr{F}(x, y, z) = \mathscr{F}(x, y, 1)$ for all non-zero $z \leqslant 1$, since $\mathscr{F}(x, y, z)$ is a continuous function. ▮

Theorem 2.3 seems to suggest the existence of a critical surface of phase transitions given by $z = z_c(x, y)$, corresponding to an adsorption transition of the directed animal in the main diagonal. It is in fact the case that $z_c(x, y) > 1$ for all finite values of x and y; the proof of this is presented next. A key ingredient in this proof will be the *density function* of visits for this model. In particular, at fixed x and y define the (microcanonical) density function of visits by

$$\log \mathscr{P}(x, y; \epsilon) = \inf_{z>0} \{\mathscr{F}(x, y, z) - \epsilon \log z\}, \tag{2.15}$$

and in this model, $\epsilon \in [0, 1/2]$. It is then the case that if

$$Z_v(x, y; \lfloor \epsilon v \rfloor) = \sum_{c, k=0}^{v} a_v(c, k, \lfloor \epsilon v \rfloor)\, x^c y^k, \tag{2.16}$$

then

$$\log \mathscr{P}(x, y; \epsilon) = \lim_{v \to \infty} \frac{1}{v} \log Z_v(x, y; \lfloor \epsilon v \rfloor), \tag{2.17}$$

and moreover

$$\mathscr{F}(x, y, z) = \sup_{0 \leqslant \epsilon \leqslant 1/2} \{\log \mathscr{P}(x, y; \epsilon) + \epsilon \log z\}. \tag{2.18}$$

Since $\mathscr{F}(x, y, z)$ is independent of z in the desorbed phase, this supremum is realised at $\epsilon = 0$ in that phase. In the adsorbed phase, the supremum is realised at values of ϵ strictly greater than zero, since $\mathscr{F}(x, y, z)$ is now a function of z, and is in fact, convex in $\log z$. Thus, the value of ϵ which realises the supremum is a non-decreasing function of $\log z$, and is in fact the energy density (of visits). Equation (2.18) also shows that

$$\log \mathscr{P}(x, y; \epsilon) \leqslant \mathscr{F}(x, y, 1), \qquad \text{for any} \quad \epsilon \in [0, 1/2]. \tag{2.19}$$

More details can be found in Janse van Rensburg;(43) see also Ellis.(24)

Lemma 2.4. For finite and fixed values of x and y, and for any value of $\epsilon \in (0, 1)$, the density function of visits obeys the following inequality:

$$\left[\frac{\epsilon^{\epsilon}\xi^{\delta}}{\delta^{\delta}(\epsilon-\delta)^{\epsilon-\delta}}\right]\mathscr{P}(x, y; \epsilon) \leqslant \mathscr{P}(x, y; \delta/(1+\delta)),$$

for any value of $\delta \in (0, \epsilon)$, and where $\xi = e^{-\mathscr{F}(x, y, 1)}/[(1+x^{-1})(1+y^{-1})]$.

Proof. Consider any positive and directed animal counted by $a_v(c, k, w)$ (see Fig. 3). This animal can be translated one step in the vertical direction to break all visits, while a single directed edge is added on the origin to keep the animal rooted there, as we did in deriving Eq. (2.14). Select u of the w vertices which were visits, and add a horizontal edge on them. This creates u new visits (and thus $u+1$ new vertices, if the new root is also counted), and may create as many as u new contacts and/or cycles. The construction can be reversed by deleting the u new visits and translating the animal back into the main diagonal. Consequently, the following inequality is found:

$$\binom{w}{u} a_v(c, k, w) \leqslant \sum_{i, j=0}^{u} a_{v+1+u}(c+i, k+j, u).$$

Choose $w = \lfloor \epsilon v \rfloor$ and $u = \lfloor \delta v \rfloor$, where $0 < \delta < \epsilon$; multiply the above by $x^c y^k$ and execute the sums over c and k. After simplification, the result is that

$$\binom{\lfloor \epsilon v \rfloor}{\lfloor \delta v \rfloor} Z_v(x, y; \lfloor \epsilon v \rfloor) \leqslant \sum_{i, j=0}^{\lfloor \delta v \rfloor} Z_{v+1+\lfloor \delta v \rfloor}(x, y; \lfloor \delta v \rfloor)\, x^{-i} y^{-j}.$$

Define $\xi_0 = (1+x^{-1})^{-1}(1+y^{-1})^{-1}$; one can show that $\sum_{i, j=0}^{\lfloor \delta v \rfloor} x^{-i} y^{-j} \leqslant \xi_0^{-\lfloor \delta v \rfloor}$. Then

$$\binom{\lfloor \epsilon v \rfloor}{\lfloor \delta v \rfloor} Z_v(x, y; \lfloor \epsilon v \rfloor) \leqslant \xi_0^{-\lfloor \delta v \rfloor} Z_{v+1+\lfloor \delta v \rfloor}(x, y; \lfloor \delta v \rfloor).$$

Take the $1/v$-power of this, and let $v \to \infty$. The result is

$$\left[\frac{\epsilon^{\epsilon}\xi_0^{\delta}}{\delta^{\delta}(\epsilon-\delta)^{\epsilon-\delta}}\right]\mathscr{P}(x, y; \epsilon) \leqslant [\mathscr{P}(x, y; \delta/(1+\delta))]^{1+\delta}.$$

Use Eq. (2.19) in the above to see that

$$[\mathscr{P}(x, y; \delta/(1+\delta))]^{1+\delta} \leqslant \mathscr{P}(x, y; \delta/(1+\delta))\, e^{\mathscr{F}(x, y, 1)\,\delta}.$$

Substitute this into the previous inequality and absorb the factor $e^{-\mathscr{F}(x, y, 1)}$ into ξ_0 to complete the proof. ▮

One can show that the maximum of the left hand side of the inequality in Lemma 2.4 occurs when $\delta = \epsilon\xi/(1+\xi) < \epsilon$, in which case the inequality becomes

$$(1+\xi)^{\epsilon}\, \mathscr{P}(x, y; \epsilon) \leqslant \mathscr{P}(x, y; \epsilon\xi_*), \tag{2.20}$$

where $\xi_* = \xi/(1+\xi+\epsilon\xi) \leqslant \xi$. Suppose that z is so large that the adsorbed phase of the positive and directed animal is encountered. This implies that the supremum in Eq. (2.18) is realized at a value of ϵ which is strictly positive, since ϵ is the density of visits in the animal.[4] Let this value of ϵ be denoted ϵ_*, then

$$\mathscr{F}(x, y, z) = \log[\mathscr{P}(x, y; \epsilon_*)\, z^{\epsilon_*}], \tag{2.21}$$

and notice that ϵ_* is the energy density (density of visits) in the model. Using Eq. (2.20) in (2.21) and observe that

$$\mathscr{F}(x, y, z) \leqslant \log\left[\mathscr{P}(x, y; \epsilon_*\xi_*)\, z^{\epsilon_*\xi_*}\, \frac{z^{\epsilon_*(1-\xi_*)}}{(1+\xi)^{\epsilon_*}}\right]. \tag{2.22}$$

Since ϵ_* is that value of ϵ which realizes the supremum in Eq. (2.18), a contradiction is found in Eq. (2.22) whenever $z^{1-\xi_*} < (1+\xi)$, *unless* $\epsilon_* = 0$. In other words, if $z^{1-\xi_*} < (1+\xi)$ then the desorbed phase of the model is encountered. Thus, if $\epsilon_* > 0$, then $\log z \geqslant [\log(1+\xi)]/(1-\xi_*)$, and if $\epsilon_* \to 0^+$, then the lower bound on the critical value of z becomes

$$\log z_c(x, y) \geqslant (1+\xi)\log(1+\xi) > 0, \tag{2.23}$$

so that the following theorem has been established:

Theorem 2.5. For all finite and non-zero values of x and y it is the case that $z_c(x, y) > 1$. ▮

---

[4] Notice that in the desorbed phase the supremum in Eq. (2.18) is found when $\epsilon = 0$, since $\mathscr{F}(x, y, z)$ is independent of z. On the other hand, since $\mathscr{F}(x, y, z)$ is convex in $\log z$, its derivative to $\log z$ exists almost everywhere, and wherever it exists, it is given by that value of ϵ which realizes the supremum.

Thus, $\mathscr{F}(x, y, z)$ is independent of z for all $0 < z < z_c(x, y)$, that is, $\mathscr{F}(x, y, z) = \mathscr{F}(x, y, 1)$ for all $z \in [0, z_c(x, y)]$. In Theorem 2.6 we show that if $\mathscr{F}(x, y, z)$ is a non-analytic function of x (or of y) for some value of $z \in [0, z_c(x, y)]$, then $\mathscr{F}(x, y, z)$ is a non-analytic function of x (or of y respectively) for all $z \in [0, z_c(x, y)]$.

Theorem 2.6. Assume that $\mathscr{F}(x, y, z)$ is non-analytic at a point (x_0, y_0, z_0) as a function of x (as a function of y) and where $z_0 < z_c(x_0, y_0)$. Assume also that the phase boundary $z_c(x, y)$ is continuous at $(x, y) = (x_0, y_0)$. Then $\mathscr{F}(x, y, z)$ is non-analytic at $(x, y) = (x_0, y_0)$ for every $z \in [0, z_c(x_0, y_0))$.

Proof. Let $\epsilon > 0$ and choose z_ϵ by

$$z_\epsilon < \inf\{z_c(x, y) \mid (x, y) \in [x_0 - \epsilon, x_0 + \epsilon] \times [y_0 - \epsilon, y_0 + \epsilon]\}.$$

$\mathscr{F}(x, y, z)$ is analytic in z for all such z_ϵ, provided that $(x, y) \in [x_0 - \epsilon, x_0 + \epsilon] \times [y_0 - \epsilon, y_0 + \epsilon]$ (if not, then there is a phase boundary $z_\delta(x, y) < z_c(x, y)$, or then $z_c(x, y)$ is not continuous; this is a contradiction). If $\epsilon \to 0$, then z_ϵ can be taken arbitrarily close to $z_c(x, y)$ since $z_c(x, y)$ is continuous. This shows that $\mathscr{F}(x, y, z)$ is independent of z for all $z < z_c(x_0, y_0)$, and a non-analyticity at (x_0, y_0, z_0) will persist for all $z_0 < z_c(x, y)$. ▌

3. COLLAPSE IN POSITIVE AND DIRECTED ANIMALS

The existence of a collapse transition in models of animals have been demonstrated using a connection with percolation models.(19) In particular, there is a point in the phase diagram of (ordinary) lattice animals with cycle and contact activities which corresponds to the critical percolation point,(29, 30) and it is also known that there are other non-analytic points in the phase diagram of animals with both cycle and contact activities.(42)

In this section we first consider collapse in directed animals rooted at the origin, but not confined to the principle wedge. This will show that there is a collapse transition for positive directed animals when the adsorption activity $z = 1$. By Theorems 2.3 and 2.6, this suggests the existence of a critical surface of collapse transitions in the phase diagram, for all those values of z corresponding to a desorbed phase on the critical surface. The method of proof is based on the fundamental theorem of percolation (as modified for directed percolation).

We next apply these techniques directly to animals in the principle wedge, and show that there is a collapsed phase in this model at some value of $z > 1$, and in fact, this phase persists even if the adsorption activity z approaches infinity (while at least one of the cycle activity x, or the contact activity y, also approaches infinity). Thus, for any $z > 1$ there is a transition into the collapsed phase.

3.1. Collapse in Directed Animals

In this section we shall consider a model of directed animals with source at the origin, but not confined to the principle wedge. As in Section 2, the animal will be oriented so that the main diagonal is horizontal, and a lexicographic ordering will again define a bottom vertex (which is the source), and a top vertex.

Define $A_v(c, k)$ to be the number of directed animals with a single source (root) at the origin, and with v vertices, c cycles and k contacts (defined as in Section 2). It is clearly the case that

$$A_v(c, k) \geqslant \sum_{w \geqslant 0} a_v(c, k, w), \tag{3.1}$$

but the first goal in this section would be to show that the free energies of directed animals, and of positive and directed animals, are in fact the same if the contact activity $z = 1$. To see this, we shall use a "*most popular*" argument, similar to the argument used in Section 2.1. Let α be any animal counted by $A_v(c, k)$. The vertices in α can be ordered in the canonical lexicographic way, and so there will be a *bottom vertex* (the lexicographic least vertex, which in this case is the origin), and a *top vertex* (which is the lexicographic most vertex). The partition function for these animals is

$$Y_v(x, y) = \sum_{c, k} A_v(c, k)\, x^c y^k. \tag{3.2}$$

Animals counted by $A_v(c, k)$ can be concatenated by choosing that vertex with first coordinate exactly one larger than the first coordinate of the top vertex of one animal to be the root (origin) of a second animal. Since this process creates no new cycles or contacts, it gives the inequality

$$\sum_{c_1, k_1} A_{v_1}(c_1, k_1)\, A_{v_2}(c - c_1, k - k_1) \leqslant A_{v_1 + v_2}(c, k). \tag{3.3}$$

Multiplying by $x^c y^k$ and summing over c and k gives the following supermultiplicative inequality for $Y_v(x, y)$:

$$Y_{v_1}(x, y)\, Y_{v_2}(x, y) \leqslant Y_{v_1 + v_2}(x, y). \tag{3.4}$$

Thus, since the number of animals grows exponentially,(47) there is a limiting free energy in this model,(39) which we define by

$$\mathcal{G}(x, y) = \lim_{v \to \infty} \frac{1}{v} \log Y_v(x, y) \tag{3.5}$$

and we can show that it is convex in each of its arguments, and therefore continuous and also differentiable almost everywhere.

We now make a connection between $\mathcal{G}(x, y)$, and the limiting free energy $\mathcal{F}(x, y, 1)$ of attached, positive and directed animals confined to the principle wedge. Since we are not interested in the adsorption of the directed animals in this section, we put $z = 1$ in Eq. (2.2) and sum over w. Let $B_v(c, k) = \sum_w a_v(c, k, w)$ be the number of attached, positive and directed animals with a single source at the origin, and with v vertices, c cycles and k contacts. Define $B_v(c, k; [h_b h_t])$ to be the set of all attached, positive and directed animals, with v vertices, c cycles, k contacts, and with bottom and top vertices which have heights equal to h_b or h_t (see Fig. 2). Note that $0 \leqslant h_b, h_t \leqslant v$. Any directed animal counted by $B_v(c, k; [h_b h_t])$ can be translated to have its source at the origin. On the other hand, any animal counted by $A_v(c, k)$ can be translated to be attached and positive and so is counted by $B_v(c, k; [h_b h_t])$, or by $B_v(c, k; [(h_b + 1)\, h_t])$ for some value of h_b. This gives

$$A_v(c, k) = 2 \sum_{h_b, h_t} B_v(c, k; [h_b h_t]). \tag{3.6}$$

Furthermore, one can define the partition function

$$X_v(x, y; [h_b h_t]) = \sum_{c, k} B_v(c, k; [h_b h_t])\, x^c y^k, \tag{3.7}$$

and observe that there are most popular values for h_b and h_t, say $[h_b^* h_t^*]$ (dependent on v, x and y), so that

$$X_v(x, y; [h_b h_t]) \leqslant X_v(x, y; [h_b^* h_t^*]). \tag{3.8}$$

Comparison to Eq. (3.6) and to Eq. (3.2) shows that

$$2X_v(x, y; [h_b^* h_t^*]) \leqslant Y_v(x, y) \leqslant 2v^2 X_v(x, y; [h_b^* h_t^*]), \tag{3.9}$$

so that $\lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_b^* h_t^*]) = \mathcal{G}(x, y)$ by Eq. (3.5).

The next lemma is necessary to complete a proof that $\mathscr{F}(x, y, 1) = \mathscr{G}(x, y)$.

Lemma 3.1. If $[h_b^* h_t^*]$ are the most popular values of $[h_b h_t]$ in $X_v(x, y; [h_b h_t])$, then

$$\mathscr{G}(x, y) = \lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_b^* h_t^*]) = \lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_b^*]).$$

Proof. It follows from Eq. (3.5) and Eq. (3.9) that

$$\mathscr{G}(x, y) = \lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_b^* h_t^*]).$$

To show the remaining equality, we shall argue as in Lemma 2.1, but there are some technical difficulaties that will be encountered. In particular, it is not trivial to see that $\lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_b^*])$ exists, and some effort is required to prove its existence. If the bottom vertex of an animal counted by $B_{v_2}(c - c_1, k - k_1; [h_2 h_3])$ is placed so that its first coordinate is a distance of $\sqrt{2}$ bigger than the top vertex of an animal counted by $B_{v_1}(c_1, k_1; [h_1 h_2])$, then they can be concatenated into a single animal by inserting a single new vertex between them. This gives the following inequality:

$$\sum_{c_1, k_1} B_{v_1}(c_1, k_1; [h_1 h_2])\, B_{v_2}(c - c_1, k - k_1; [h_2 h_3]) \leqslant B_{v_1 + v_2 + 1}(c, k; [h_1 h_3])$$

with the result that after multiplication with $x^c y^k$ and summing over c and k, we obtain the supermultiplicative relation

$$X_{v_1}(x, y; [h_1 h_2])\, X_{v_2}(x, y; [h_2 h_3]) \leqslant X_{v_1 + v_2 + 1}(x, y; [h_1 h_3])$$

for $X_v(x, y; [h_1 h_2])$: In particular, the limit $\lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [hh])$ exists.

Define $[h_b^* h_t^*]$ to be the most popular values of $[h_b h_t]$ in $X_v(x, y; [h_b h_t])$. The concatenation above can be extended to alternate between directed animals counted by $X_v(x, y; [h_b^* h_t^*])$ and $X_v(x, y; [h_t^* h_b^*])$ to see that

$$[X_v(x, y; [h_b^* h_t^*])]^M\, [X_v(x, y; [h_t^* h_b^*])]^{M-1} \leqslant X_{(2M-1)(v+1)-1}(x, y; [h_b^* h_t^*]);$$

$$[X_v(x, y; [h_t^* h_b^*])]^M\, [X_v(x, y; [h_b^* h_t^*])]^{M-1} \leqslant X_{(2M-1)(v+1)-1}(x, y; [h_t^* h_b^*]).$$

Take the logarithms of these, divide by $(2M-1)(v+1)-1$, and take M to infinity, such that the lim infs on the right hand sides of the above inequalities are found. This shows that

$$\frac{1}{2(v+1)}\log X_v(x,y;[h_b^*h_t^*])+\frac{1}{2(v+1)}\log X_v(x,y;[h_t^*h_b^*])$$

$$\leqslant \liminf_{M\to\infty}\frac{1}{(2M-1)(v+1)-1}\log X_{(2M-1)(v+1)-1}(x,y;[h_b^*h_t^*])=\mathscr{G}(x,y);$$

$$\frac{1}{2(v+1)}\log X_v(x,y;[h_t^*h_b^*])+\frac{1}{2(v+1)}\log X_v(x,y;[h_b^*h_t^*])$$

$$\leqslant \liminf_{M\to\infty}\frac{1}{(2M-1)(v+1)-1}\log X_{(2M-1)(v+1)-1}(x,y;[h_t^*h_b^*]).$$

Now take the lim sup on the left hand sides of these inequalities by taking $v\to\infty$, while replacing the lim inf on the right hand sides by a lim sup. This shows that

$$\limsup_{v\to\infty}\frac{1}{v}\log X_v(x,y;[h_t^*h_b^*])=\mathscr{G}(x,y). \tag{‡}$$

It now only remains to show that the limit exists. To see that, define

$$\liminf_{v\to\infty}\frac{1}{v}\log X_v(x,y;[h_t^*h_b^*])=\mathscr{K}(x,y).$$

Let $\{v_i\}$ be any subsequence of the integers such that

$$\lim_{i\to\infty}\frac{1}{v_i}\log X_{v_i}(x,y;[h_t^*h_b^*])=\mathscr{K}(x,y).$$

Define M_i to be the largest integer, dependent on v, such that $M_i(v+1)+l_i=v_i$, for each i, where $3(v+1)\leqslant l_i<4(v+1)$, and where v is fixed large enough so that $X_{l_i}(x,y;[h_t^*h_b^*])\neq 0$ for any such l_i. Then it follows from concatenation that

$$[X_v(x,y;[h_t^*h_t^*])]^{M_i}X_{l_i}(x,y;[h_t^*h_b^*])\leqslant X_{v_i}(x,y;[h_t^*h_b^*]).$$

Take logarithms of this inequality, divide by v_i, and take $v_i \to \infty$ by taking $i \to \infty$ with v fixed. Then $M_i \to \infty$ and

$$\frac{1}{v+1} \log X_v(x, y; [h_t^* h_t^*]) \leqslant \liminf_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_b^*]). \qquad (\dagger)$$

Now take $v \to \infty$ to see that

$$\lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_t^*]) \leqslant \liminf_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_b^*]),$$

where the concatenation of directed animals counted by $B_v(c, k; [h_1 h_2])$ above shows that the last limit exists. This shows that $\lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_t^*]) - \liminf_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_b^*])$ is not positive. Next, concatenating again directed animals counted by $B_v(c, k; [h_1 h_2])$, we obtain

$$X_v(x, y; [h_t^* h_b^*])\, X_v(x, y; [h_b^* h_t^*]) \leqslant X_{2v+1}(x, y; [h_t^* h_t^*]).$$

Taking logarithms of this, dividing by v, and taking the lim inf on the left hand side, shows that

$$\mathscr{G}(x, y) + \liminf_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_b^*]) \leqslant 2 \lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_t^*]).$$

Since $\lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_t^*]) - \liminf_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_b^*])$ is never positive, we have

$$\mathscr{G}(x, y) \leqslant \lim_{v \to \infty} \frac{1}{v} \log X_v(x, y; [h_t^* h_t^*]).$$

Comparing this to equations (†) and (‡) above completes the proof. ∎

We would in fact like to show that $\mathscr{G}(x, y) = \mathscr{F}(x, y, 1)$, the free energy of positive directed animals with adsorption activity $z = 1$. To do this, we concatenate animals counted by $X_v(x, y; [h_b^* h_t^*])$ and $X_v(x, y; [h_t^* h_b^*])$ in a 45° wedge. This is done in Fig. 4, and if k is large enough, then the directed animal is confined in the principle wedge. The result is that (if $2M$ animals are concatenated)

$$Z_{2M(v+1)+k-1}(x, y, 1) \geqslant [X_v(x, y; [h_b^* h_t^*])]^M\, [X_v(x, y; [h_t^* h_b^*])]^M. \qquad (3.10)$$

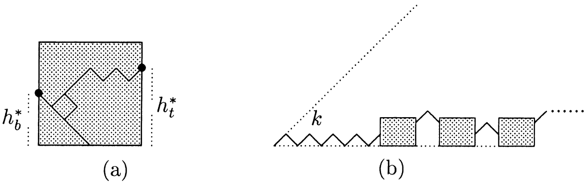

Fig. 4. (a) A schematic drawing of an animal counted by the partition function $X_v(x, y; [h_b^* h_t^*])$. (b) These animals can be concatenated alternatingly from the partition functions $X_v(x, y; [h_b^* h_t^*])$ and $X_v(x, y; [h_t^* h_b^*])$ to create a directed animal in a 45° degree wedge, provided that k, the number of edges which joins the animal to the origin, is large enough ($k > v$ is sufficient).

On the other hand, it also follows from Eqs. (3.6) and (3.7) that

$$Z_v(x, y, 1) \leqslant X_v(x, y; [h_b^* h_t^*]). \tag{3.11}$$

Equations (3.10) and (3.11) finally gives the desired result:

Theorem 3.2.

$$\mathscr{F}(x, y, 1) = \mathscr{G}(x, y).$$

Proof. The proof follows by taking the logarithm of Eq. (3.10), dividing by $2M(v+1)+k-1$ and taking $M \to \infty$ (with v fixed). This gives

$$\mathscr{F}(x, y, 1) \geqslant \frac{1}{2(v+1)} [\log X_v(x, y; [h_b^* h_t^*]) + \log X_v(x, y; [h_t^* h_b^*])].$$

The variable k has now disappeared, as has any reference to the wedge in Fig. 4. Thus, we can take $v \to \infty$ safely to obtain $\mathscr{F}(x, y, 1) \geqslant \mathscr{G}(x, y)$, using Lemma 3.1. On the other hand, the opposite inequality can be obtained from Eq. (3.11). ▮

To show that there is a collapse transition in the positive directed animal at $z = 1$, we must only show that $G(x, y)$ is non-analytic in its arguments. This we do by considering the relation of a directed animal rooted at the origin in the first quadrant to the process of directed percolation.

Define a directed site-bond percolation process on the edges and vertices of the first quadrant in the directed square lattice by introducing the following probabilities. Any edge (bond) is open with probability p_b, and closed with probabality $q_b = 1 - p_b$. Every vertex (site) is occupied with probability p_s, and not occupied with probability $q_s = 1 - p_s$. Grow a directed percolation cluster from the origin as follows: If the origin is occupied, then it is the first site in the cluster. If not, then the cluster at the origin has size zero. Next, consider sites in the perimeter of the current cluster. If such sites are occupied, *and* they can be reached from the current cluster via an open and directed edge, then they become part of the current cluster. This is continued until the process terminates. The result is a directed animal, with a single source at the origin.

Every directed animal at the origin in a process of directed percolation has a certain number of properties, and we enumerate them here:

$$\begin{aligned} v&: \text{ total number of vertices, including the root and visits;}\\ v_1&: \text{ the number of vertices with 1 predecessor in the animal;}\\ v_2&: \text{ the number of vertices with 2 predecessors in the animal;}\\ n&: \text{ the number of edges in the animal;}\\ c&: \text{ the number of cycles in the animal;}\\ k&: \text{ the number of contacts in the animal;}\\ s_1&: \text{ the number of perimeter sites incident with one perimeter edge;}\\ s_2&: \text{ the number of perimeter sites incident with two perimeter edges.}\end{aligned} \tag{3.12}$$

Every occupied vertex Y in the directed cluster at the origin, except for the root at the origin, has a *predecessor* X (which is also occupied, and in the cluster) such that XY is an open edge from X to Y. Since each vertex, except the root, has a predecessor, it is the case that $v_1 + v_2 = v - 1$. More such relations can be derived between the numbers in (3.12); in particular, one may show that

$$\begin{aligned} v_1 &= v - c - 1;\\ v_2 &= c;\\ s_1 &= v - 2k + c + 1;\\ s_2 &= k - c.\end{aligned} \tag{3.13}$$

To see these, notice that each cycle can only be formed when a vertex has two predecessors and *vice versa*, so $v_2 = c$, and therefore $v_1 = v - 1 - c$. If a site has two predecessors in the animal, then there is a contact between the predecessors. Since such a site is either occupied, in which case it is a cycle, or not, in which case it is counted by s_2, thus it follows that $s_2 = k - c$. Note also that $2v = n + s_1 + 2s_2$, since the two edges pointing to descendants of a vertex in the animal are either edges in the animal, or points to perimeter sites. Since $v-n+c=1$, this shows that $s_1=v-2k+c+1$.

Suppose that the directed animal α is obtained as the cluster at the origin in a directed percolation process. The probability that α occurs is given by the probability that every vertex in α is occupied, that there is a directed open path from the origin to every vertex in α, and that the perimeter of α occurs (and is vacant). Every vertex in α, except the root, has one occupied predecessor with probability p_b, and two occupied predecessors with probability $1-q_b^2$ (since one open edge from either predecessor will be enough to add the vertex to the cluster). A perimeter site has one occupied predecessor with probability $1-p_s p_b$ (it can only be a vertex in the animal if it is occupied and the edge from its occupied predecessor is open), and two occupied predecessors with probability $q_s+p_s q_b^2$ (that is, it is itself not occupied, or it is occupied, but both edges into it are closed). Taken together, this shows that

$$\begin{aligned} P_v(\alpha) &= p_s^v p_b^{v_1} (1-q_b^2)^{v_2} \, (1-p_s p_b)^{s_1} \, (q_s+p_s q_b^2)^{s_2}; \\ &= \left[\frac{1-p_s p_b}{p_b}\right] [p_s p_b (1-p_s p_b)]^v \left[\frac{(1-q_b^2)(1-p_s p_b)}{p_b(q_s+p_s q_b^2)}\right]^c \left[\frac{q_s+p_s q_b^2}{(1-p_s p_b)^2}\right]^k . \end{aligned} \tag{3.14}$$

To simplify this equation, define

$$x=\frac{(1-q_b^2)(1-p_s p_b)}{p_b(q_s+p_s q_b^2)}, \qquad y=\frac{q_s+p_s q_b^2}{(1-p_s p_b)^2}, \tag{3.15}$$

to be the weights of cycles and contacts in the directed cluster. The probability that the directed animal at the origin has exactly v vertices is then given by

$$P_v(p_s, p_b)=\sum_{\alpha} P_v(\alpha)=\left[\frac{1-p_s p_b}{p_b}\right] [p_s p_b(1-p_s p_b)]^v \sum_{c,k} A_v(c,k)\, x^c y^k. \tag{3.16}$$

Taking logarithms, dividing by v and taking $v\to\infty$ gives

$$\lim_{v\to\infty} \frac{1}{v} \log P_v(p_s, p_b) = -\zeta(x,y) = \log(p_s p_b(1-p_s p_b)) + \mathcal{G}(x,y). \tag{3.17}$$

where the existence of the limit is gauranteed by the existence of the limit defining $\mathcal{G}(x,y)$ in Eq. (3.5). This result ties the limiting free energy, and thus the phase behaviour, of directed animals to the process of directed percolation.

The fundamental theorem of percolation states that there are non-trivial values of p_s and p_b where percolation occurs; that is, where the cluster at the origin grows to infinity with non-zero probability. It can also be demonstrated that $P_v(p_s, p_b)$, defined in Eq. (3.16), has specific asymptotic behaviour in the percolated and in the unpercolated phases. We state this, without proof,[5] in Theorem 3.3.

Theorem 3.3. The probability $P_v(p_s, p_b)$ that the directed cluster at the origin contains v vertices is related to $\zeta(x, y)$ as follows:

$$P_v(p_s, p_b) = e^{-\zeta(x, y)\, v + o(v)}.$$

Moreover, if both $p_s < 1$ and $p_b < 1$ are close to 1, then $\zeta(x, y) = 0$ and the system percolates. Otherwise, $\zeta(x, y) > 0$ and the system is in the sub-critical phase. Hence $\zeta(x, y)$ is a non-analytic function of its arguments. ▮

Consider x and y as functions of p_s and p_b, as in Eq. (3.15). In Fig. 5 we plot x against p_s and p_b, and in Fig. 6 we plot y against p_s and p_b. The important observation in these figures is that in the first case

$$\begin{aligned} &\lim_{p_b \to 1^-} [\lim_{p_s \to 1^-} x(p_s, p_b)] = \infty, \\ &\lim_{p_s \to 1^-} [\lim_{p_b \to 1^-} x(p_s, p_b)] = 1, \end{aligned} \tag{3.18}$$

and in the second case

$$\begin{aligned} &\lim_{p_b \to 1^-} [\lim_{p_s \to 1^-} y(p_s, p_b)] = 1, \\ &\lim_{p_s \to 1^-} [\lim_{p_b \to 1^-} y(p_s, p_b)] = \infty. \end{aligned} \tag{3.19}$$

Thus, either x, or y, or both, approach ∞ if $(p_s, p_b) \to (1^-, 1^-)$. Indeed, if we should suppose that $q_b = \alpha q_s$ with $\alpha \in (0, \infty)$, then $xy = \frac{1+\alpha q_s}{q_s(1+\alpha p_s)}$. Hence $xy \to \infty$ as $q_s \to 0^+$ and so either $x \to \infty$ or $y \to \infty$, or both.

---

[5] The proof that $\zeta(x, y) = 0$ in the percolated phase as stated in Theorem 3.3 is in fact not dissimilar to the proof of Theorem 3.6. The proof that $\zeta(x, y) > 0$ in the unpercolated phase is lengthy, but similar to the proof for ordinary bond percolation, see for example Grimmett.(33)

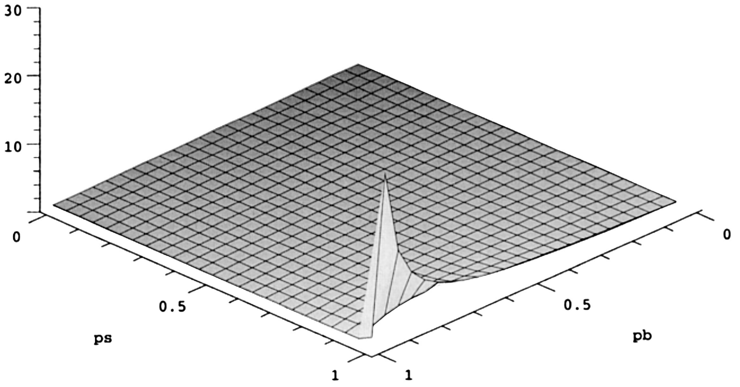

Fig. 5. $x(p_s, p_b)$ plotted against p_s and p_b.

More generally, let $q_b = f_b(t)$ and $q_s = f_s(t)$ be parametric equations of a continuous curve in the unit square with with endpoints $(0, 0)$ and $(1, 1)$ (so that $t \in [0, 1]$ and $f_b(0) = f_s(0) = 1$ and $f_b(1) = f_s(1) = 0$). Then it follows that $xy = (1 + f_b(t))/(f_s(t) + f_b(t) - f_b(t)\, f_s(t))$, and $xy = 1$ if $t = 0$ and $xy = \infty$ if $t = 1$. In other words, if the point $(p_s, p_b) = (1, 1)$ is approached along any continuous curve, then $xy \to \infty$. Theorem 3.3 can be applied in these circumstances to conclude that $\zeta(x, y) = 0$, if (p_s, p_b) is close enough to $(1, 1)$. That is, there is a region in the unit square, which has the point $(1, 1)$ in its interior, where percolation has taken place, and where the limiting free energy of the animal model is known explicitly (and interpreted to have undergone a collapse transition). Outside this region

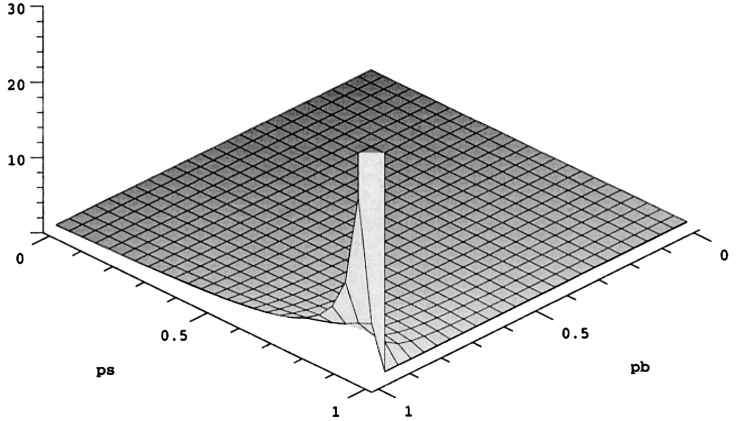

Fig. 6. $y(p_s, p_b)$ plotted against p_s and p_b.

is the unpercolated phase, corresponding to the expanded phase in the directed animal (notice that x and y are small if both p_s and p_b are small). It follows from Eq. (3.17) that if (p_s, p_b) is close enough to (1, 1), then

$$\mathscr{G}_\zeta(x, y) = \mathscr{F}(x, y, 1) = -\log[p_s p_b(1 - p_s p_b)], \tag{3.20}$$

and the expressions in Eq. (3.15) can in principle be inverted to write an explicit expression for $\mathscr{F}(x, y, 1)$ in the collapsed phase. On the other hand, if (p_s, p_b) approach (0, 0), then Theorem 3.2 gives $\zeta(p_s, p_b) > 0$, and so $\mathscr{F}_\zeta(x, y, 1) \neq -\log[p_s p_b(1 - p_s p_b)]$ in that case. By the arguments above, there is a non-analyticity in $\mathscr{F}_\zeta(x, y, 1)$ along any continuous curve which joints the points (0, 0) and (1, 1), and the locus of the non-analyticities are interpreted as a phase boundary.

There is a phase boundary separating the collapsed phase from the expanded phase. To see this, argue as follows. Let L be the set of all non-analyticities in $\mathscr{G}_\zeta(x, y)$ in the closed unit square $S = [0, 1] \times [0, 1]$ (since (p_s, p_b) takes values in the unit square). This set is not necessarily connected, but it must have a component which separates the point (0, 0) from the point (1, 1) in S: If $S \setminus L$ has a path-component containing both the points (0, 0) and (1, 1), then there is a continuous curve with endpoints in (0, 0) and (1, 1) and which is contained in $S \setminus L$. But since every continuous function must pass through a non-analyticity in L, this is a contradiction, and so we conclude that $S \setminus L$ has at least two path-components, with (0, 0) and (1, 1) in two different such components. We interprete the boundary of the component in $S \setminus L$ containing the point (1, 1) as the phase boundary.

We have now established that $\mathscr{F}(x, y, z)$ has a phase boundary in the plane $z = 1$, but by Theorem 2.6 this immediately extends to a critical surface for all $z \leqslant z_c(x, y)$, where $z_c(x, y) > 1$ is the critical adsorption activity. Moreover, this critical surface is constant in the z-directions (independent of z). The meeting between this critical surface, and the critical surface of adsorption transitions, is presumably a line of triple points where expanded, collapsed and adsorbed directed animals coexist.

3.2. Collapse in Positive Directed Animals with $z > 1$

In this section we examine the existence of a collapsed phase for attractive values of the adsorption activity z; that is, for values of z greater than 1. As in Section 3.1, we shall again rely on a connection to directed percolation, but in this case the directed percolation will be confined to the principle wedge Λ.

In (3.12) we have defined a set of quantities associated with a directed animal; this list will not be adequate for directed percolation in Λ.

As before, v, n, c and k will still be the number of vertices, edges, cycles and contacts in a directed animal rooted at the origin, and confined to Λ. To this list, we add the following:

$$\begin{aligned}
v_1&: \text{ the number of vertices with one predecessor, excluding visits;}\\
v_2&: \text{ the number of vertices with two predecessors;}\\
w&: \text{ the number of vertices (excluding the root) lying on the main diagonal;}\\
w_s&: \text{ the number of vertices in the super-diagonal (the line } y=x+1\text{);}\\
s_d&: \text{ the number of perimeter sites in the main diagonal;}\\
s_1&: \text{ the number of perimeter sites incident with one perimeter edge;}\\
s_2&: \text{ the number of perimeter sites incident with two perimeter edges.}
\end{aligned} \tag{3.21}$$

The definition of s_1 above is interpreted as the number of perimeter sites incident with one perimeter edge *and not* in the main diagonal. Observe that $s_d+s_1+2s_2=q$, where q is the number of perimeter edges defined in Eq. (2.1) and where we have used the fact that each perimeter site in the main diagonal is incident with exactly one perimeter edge).

The following relations can be derived amongst the quantities in (3.21):

$$\begin{aligned}
v_1 &= v-w-c-1;\\
v_2 &= c;\\
s_1 &= v-2k+c-w_s;\\
s_2 &= k-c;\\
s_d &= w_s-w.
\end{aligned} \tag{3.22}$$

As before, it is still the case that $v_2=c$, and $n+1=v+c$ (Eq. (2.1)) while $v_1+v_2+w=v-1$. Thus $v_1=v-1-w-c$. Every contact consists of two occupied sites with a single common descendant. If this descendant is occupied, then a cycle is formed, if not, then it is a perimeter site with two predecessors. Thus $k=c+s_2$. Since the source has no predecessors, it is the case that $w_s=s_d+w$. Finally, the number of perimeter edges is $q=s_1+s_d+2s_2$, and from Eq. (2.1) we obtain that $s_1=v-w-c-s_d-2s_2$. Substitution of s_d and s_2 gives the required expression for s_1.

Define the following percolation process on Λ. Any edge is open with probability p_b, and closed with probability $q_b=1-p_b$. Every vertex (site) in Λ, including the origin, but excluding the remaining vertices on the main diagonal, are occupied with probability p_s, and otherwise not occupied.

Lastly, vertices in the main diagonal, except the origin, are occupied with probability p_w, and otherwise not occupied. The cluster at the origin is now grown as before, and it is a directed (site)-animal with source at the origin, and confined to the wedge Λ.

Consider now the probability $P_v(\alpha)$ that the animal incident with the origin is the animal α with exactly v vertices (and n edges, w visits, w_s visits to the superdiagonal, c cycles and k contacts). The animal α occurs if every vertex in α is occupied, and if there is an open path (of edges and occupied vertices) from the origin to every vertex in α, and if α's perimeter occurs. A vertex in α, which is neither the root nor a visit, has one occupied predecessor with probability p_b (since the edge must be open to add the vertex to the cluster), and two occupied predecessors with probability $1-q_b^2$ (since one open edge from either predecessor will be enough the add the vertex to the cluster). A diagonal site is in the perimeter with probability $(1-p_b p_w)$ (it will be in α if both the edge into it is open, and itself is occupied). Other perimeter sites have one occupied predecessor with probability $(1-p_s p_b)$ (if it is occupied and the edge from its predecessor is open, then it must be part of the cluster) and two occupied predecessors with probability $(q_s+p_s q_b^2)$ (that is, it is itself unoccupied, or it is occupied, but both the edges into it are closed). Taken together, this shows that

$$P_v(\alpha) = p_s^{v-w}\, p_w^w\, p_b^{v_1} (1-q_b^2)^{v_2}\, (1-p_s p_b)^{s_1}\, (q_s+p_s q_b^2)^{s_2}\, (1-p_b p_w)^{s_d}. \tag{3.23}$$

Substitution of the above from Eq. (3.22) leads to

$$\begin{aligned} P_v(\alpha) = {} & \left[\frac{1}{p_b}\right] [p_s p_b(1-p_s p_b)]^v \left[\frac{(1-q_b^2)(1-p_s p_b)}{p_b(q_s+p_s q_b^2)}\right]^c \left[\frac{q_s+p_s q_b^2}{(1-p_s p_b)^2}\right]^k \\ & \times \left[\frac{p_w}{p_s p_b(1-p_b p_w)}\right]^w \left[\frac{1-p_b p_w}{1-p_s p_b}\right]^{w_s}. \end{aligned} \tag{3.24}$$

We recognize x and y, defined in Eq. (3.15) in the above, and we may further define

$$z = \frac{p_w}{p_s p_b(1-p_b p_w)}, \qquad z_s = \frac{1-p_b p_w}{1-p_s p_b}, \tag{3.25}$$

for the weights of vertices in the main diagonal and in the superdiagonal respectively. Summing Eq. (3.24) over all directed animals α with v vertices and in Λ then give the probability that the cluster at the origin has v vertices:

$$P_v = \sum_{\alpha} P_v(\alpha) = \left[\frac{1}{p_b}\right] [p_s p_b(1-p_s p_b)]^v \sum_{c,k,w,w_s} a_v(c,k,w,w_s)\, x^c y^k z^w z_s^{w_s}, \tag{3.26}$$

Using the techniques in Section 2.1, it can be shown that the limit

$$\lim_{v \to \infty} \frac{1}{v} \log P_v(p_s, p_b, p_w) = -\zeta(p_s, p_b, p_w) \leqslant 0 \tag{3.27}$$

exists.

The value of $\zeta(p_s, p_b, p_w)$ in Eq. (3.27) will be of interest to us. It is related to $\zeta(p_s, p_b)$ defined in Eq. (3.17) and Theorem 3.3, and it will play a role here similar to that played by $\zeta(p_s, p_b)$ there. It is related closely to the limiting free energy in that model, and will show that the limiting free energy in a model of collapsing positive directed animals is a non-analytic function of the contact or cycle activities, *even* at values of $z > 1$. To see this, we first consider more closely the process of directed percolation in Λ with p_s, p_b and p_w defined as above.

Denote the origin (0, 0) by $\tilde{\mathbf{0}}$. Define $\theta(p_s, p_b, p_w; X)$ to be the probability that the vertex X is in an infinite cluster in the directed percolation model. Then it follows from the fundamental theorem of percolation that $\theta(p_s, p_b, p_w; \tilde{\mathbf{0}}) = 0$ in the subcritical phase (where the origin is in a finite cluster), and $\theta(p_s, p_b, p_w; \tilde{\mathbf{0}}) > 0$ in the supercritical phase. Moreover, the values of p_s and p_b which are critical with respect to percolation are not the trivial values 0 or 1. Notice also that $\theta(p_s, p_b, p_w)$ is monotone non-decreasing in its arguments p_s, p_b, or p_w.

Theorem 3.4. If we denote the vertex (0, 1) by $\tilde{\mathbf{1}}$, then $\theta(p_s, p_b\ p_s; \tilde{\mathbf{0}}) = \theta(p_s, p_b, 0; \tilde{\mathbf{1}})$.

Proof. Observe that if $p_w = 0$, then a directed percolation process in the wedge confined to the first quadrant and above the superdiagonal is obtained. All vertex probabilities are p_s, and all bond probabilities are p_b. This can be translated one step vertically to a model with $p_w = p_s$ and the theorem follows.[6] ▮

Theorem 3.5. If $0 < p_w \leqslant p_s$, then

$$\theta(p_s, p_b, p_w; \tilde{\mathbf{0}}) = 0 \quad \Leftrightarrow \quad \theta(p_s, p_b, p_s; \tilde{\mathbf{0}}) = 0.$$

Proof. This theorem follows directly from the following string of inequalities:

---

[6] The edges incident with the main diagonal, but outside the principle wedge, can be open with probability p_b, but that is irrelevent in this model. Since all the vertices below the main diagonal are unoccupied (with probability 1), there are no directed paths through these edges, and no cluster can be connected via them.

$$\begin{aligned}
&\theta(p_s, p_b, p_s; \tilde{\mathbf{0}}) \\
&\qquad \geqslant \theta(p_s, p_b, p_w; \tilde{\mathbf{0}}) && \text{since } p_s \geqslant p_w; \\
&\qquad = p_s p_b\, \theta(p_s, p_b, p_w; \tilde{\mathbf{1}}) && \text{since } \tilde{\mathbf{0}} \text{ is occupied with probability } p_s; \\
&\qquad \geqslant p_s p_b\, \theta(p_s, p_b, 0; \tilde{\mathbf{1}}) && \text{since } p_w > 0 \\
&\qquad = p_s p_b\, \theta(p_s, p_b, p_s; \tilde{\mathbf{0}}) && \text{by Theorem 3.4.} \quad \blacksquare
\end{aligned}$$

An *open path* in the directed percolation process in Λ is a sequence alternating between vertices and directed edges, starting and terminating with a vertex, and with all vertices and directed edges along it occupied and open respectively. Theorem 3.6 shows that $\zeta(p_s, p_b, p_w) = 0$ in the supercritical phase. The method of proof is similar to that of ordinary percolation,⁽³³⁾ but the directed nature of the percolation process here again produces some technical difficulties which must be overcome.

Theorem 3.6. $\theta(p_s, p_b, p_w; \tilde{\mathbf{0}}) > 0 \Rightarrow \zeta(p_s, p_b, p_w) = 0.$

Proof. Let $p \equiv \{p_s, p_b, p_w\}$ and let $\theta(p; \tilde{\mathbf{0}})$ be the probability that the origin is the source of an infinite cluster. Let m be a positive integer, and define T_m to be the triangle in Λ with corners $(0, 0)$, $(m, 0)$ and (m, m). The number of vertices in T_m is $m(m+1)/2$. The boundary of T_m will be indicated by ∂T_m and it consists of all those vertices in the set $\{(i, 0), (m, i), (i, i) \mid i = 0, 1,..., m\} \cup \{(i+1, i) \mid i = 0, 1,..., m-1\}$.

Suppose that r_m is the number of occupied vertices in T_m from which an infinite directed cluster can be grown. If X is a vertex in T_m, then X is counted by r_m with probability $\theta(p; \tilde{\mathbf{0}})$, by translating the percolation process to have origin X. We shall first find an upper bound on the expectation of r_m, and use that to find a lower bound on the probability that r_m exceeds $[\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]$.

Denote the probability of the event α by $P(\alpha)$, and notice that $r_m \leqslant m(m+1)/2$, and that the expected value of r_m is bounded above by

$$\begin{aligned}
E(r_m) = &\sum_{n=0}^{m(m+1)/2} nP(r_m = n) \\
&\leqslant [m(m+1)/2]\, P(r_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]) \\
&\quad + [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]\, P(r_m < [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]) \\
&\leqslant [m(m+1)/2]\, P(r_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]) \\
&\quad + [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]
\end{aligned}$$

But surely $E(r_m) \geqslant \theta(p; \tilde{\mathbf{0}})[m(m+1)/2]$, since each vertex X in T_m is the source of an infinite directed cluster with probability $\theta(p; 0)$, and thus it follows that

$$P(r_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]) \geqslant \theta(p; \tilde{\mathbf{0}})/2.$$

Let u_m be the number of vertices in T_m which can be reached from ∂T_m by a directed path of open edges and occupied vertices. Similarly, define v_m to be the number of vertices in T_m which is the source of an open path(s) to ∂T_m. Suppose that X is a vertex counted by v_m, and that X_p is the open directed path from ∂T_m to X. By reflecting T_m though its symmetric anti-diagonal, let T'_m be the image of T_m, X' be the image of X, and X'_p the image of X_p. Under this reflection, all open edges are reflected to open edges, closed edges to closed edges, occupied vertices to occupied vertices, and unoccupied vertices to unoccupied vertices. Observe that if the direction of edges in T'_m are reversed, then X'_p is a directed path of open edges from X' to ∂T_m. Thus, X' will be counted by u_m in T'_m. Similarly, one can establish that vertices counted by u_m in T'_m are reflected to vertices counted by v_m in T_m. Averaging over all realizations of open and closed edges, and occupied and unoccupied vertices in T_m, this shows that the most likely value of u_m and v_m are the same; let this number be M.

Notice now that $v_m \geqslant r_m$, since each vertex counted by r_m is the source of an open directed path of edges to ∂T_m. Thus

$$\begin{aligned} P(v_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]) &\geqslant P(r_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2]) \\ &\geqslant \theta(p; \tilde{\mathbf{0}})/2. \end{aligned}$$

Thus, $M = \Theta(m^2)$, provided that $\theta(p; \tilde{\mathbf{0}}) > 0$.

The event K_m that all vertices in ∂T_m are occupied, all directed edges between vertices in ∂T_m are open, and all directed edges between ∂T_m and ∂T_{m+1} are closed, has probability $p_w^{m+1} p_s^{3m-1} (1-p_b)^{m+1} p_b^{4m}$. Thus, the probability that both K_m occurs, and that $[m(m+1)/2] \geqslant v_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2] \times [m(m+1)/2]$ is at least

$$\begin{aligned} &P(r_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2])\, P(K_m) \\ &\qquad \geqslant \tfrac{1}{2}\theta(p; \tilde{\mathbf{0}})\, p_w^{m+1} p_s^{3m-1} (1-p_b)^{m+1}\, p_b^{4m}. \end{aligned}$$

The most likely value of v_m is M, and so it follows that $M \geqslant r_m$ as well. M is also the most likely value of u_m, the number of vertices in T_m which can be reached by an open path from ∂T_m, and thus from the origin. Let $P_M(p)$ be the probability that the cluster at the origin has size M vertices. Since v_m takes on at most $[m(m+1)/2]$ values, the most likely one which is M, it follows that $P_M(p) \geqslant [P(v_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2])]/[m(m+1)/2]$. Thus

$$\begin{aligned} P_M(p) \geqslant [P([m(m+1)/2] \\ &\geqslant r_m \geqslant [\theta(p; \tilde{\mathbf{0}})/2][m(m+1)/2])\, P(K_m)]/[m(m+1)/2] \\ &\geqslant \tfrac{1}{2}\theta(p; \tilde{\mathbf{0}})\, p_w^{m+1} p_s^{3m-1} (1-p_b)^{m+1}\, p_b^{4m}/[m(m+1)/2]. \end{aligned}$$

Thus, since $M = \Theta(m^2)$ if $\theta(p; \tilde{\mathbf{0}}) > 0$, it follows that $\lim_{M \to \infty} [P_M(p)]^{1/M} = 1$ in the supercritical phase. ▮

Let us now write $\theta(p_s, p_b, p_w)$ to mean $\theta(p_s, p_b, p_w; \tilde{\mathbf{0}})$. The following corollary is a consequence of Theorem 3.6.

Corollary 3.7. $\theta(p_s, p_b, p_s) > 0 \Rightarrow \zeta(p_s, p_b, p_w) = 0 \;\; \forall p_w \in (0, 1]$.

Proof. Observe that if $\theta(p_s, p_b, p_s) > 0$, then it follows also that $\theta(p_s, p_b, p_w) > 0$, for all $p_w \in (0, 1]$. This follows in the first instance from Theorem 3.5 if $p_w \leqslant p_s$, and from the fact that $\theta(p_s, p_b, p_w)$ is non-decreasing with p_w if $p_w \geqslant p_s$. The corollary is now a direct consequence of Theorem 3.6. ▮

This corollary shows that $\zeta(p_s, p_b, p_w) = 0$ in the supercritical phase, and that reducing the value of p_w is not sufficient to stop the system from percolating, and move it from the supercritical phase into the subcritical phase.

We next consider $\zeta(p_s, p_b, p_w)$ in the subcritical phase. To see that $\zeta(p_s, p_b, p_w) > 0$ if $\theta(p_s, p_b, p_w) = 0$ requires more careful arguments; here we will only provide an outline of the proof. Consider first a model of fully directed percolation in the first quadrant of the square lattice. Let the diagonal and sub-diagonal vertices be occupied with probability p_w (except for the vertices (0, 0) and (1, 0), which is occupied with probability p_s). The remaining vertices are occupied with probability p_s. Suppose that edges are open with probability p_b. Let $P_v^D(p_s, p_b, p_w)$ be the probability that the origin is in a cluster of size v vertices.

It follows that $P_v(p_s, p_b) = P_v^D(p_s, p_b, p_s)$, where P_v is defined in Eq. (3.16), and thus from Theorem 3.3 one obtains that

$$\lim_{v \to \infty} \frac{1}{n} \log P_v^D(p_s, p_b, p_w) \leqslant \lim_{v \to \infty} \frac{1}{n} \log P_v^D(p_s, p_b, p_s) < 0; \qquad \forall p_w < p_s, \tag{3.28}$$

in the subcritical phase, since the directed percolation is an increasing process with p_w. The first quadrant can be divided into two principle wedges by cutting the edges between the diagonal and the subdiagonal, the first originating at the vertex (0, 0), and the second originating at the vertex (1, 0). The probability that (0, 0) is the source of a directed cluster of size v in the first principle wedge is $P_v(p_s, p_b, p_w)$, and similarly, this is also the probability that (1, 0) is the source of a cluster of size v in the second principle wedge. Thus, joining these clusters with the edge between (0, 0) and (1, 0), we get

$$P^D_{2v}(p_s, p_b, p_w) \geqslant p_b[P_v(p_s, p_b, p_w)]^2. \tag{3.29}$$

Taking logarithms, dividing by $2n$ and letting $n \to \infty$, shows that $\zeta(p_s, p_b, p_w) > 0$ in the subcritical phase and the following theorem is obtained:

Theorem 3.8. $\theta(p_s, p_b, p_w) = 0 \Rightarrow \zeta(p_s, p_b, p_w) > 0$, and moreover, if $0 < p_w \leqslant p_s$ then $\zeta(p_s, p_b, p_s) > 0 \Leftrightarrow \zeta(p_s, p_b, p_w) > 0$.

Proof. That $\theta(p_s, p_b, p_w) = 0 \Rightarrow \zeta(p_s, p_b, p_w) > 0$ follows from the arguments above. To prove the rest of the theorem, consider the following:

$$\begin{aligned} \zeta(p_s, p_b, p_w) > 0 &\Rightarrow \theta(p_s, p_b, p_s) = 0, && \text{by Corollary 3.7;} \\ &\Rightarrow \zeta(p_s, p_b, p_s) > 0, && \text{by the above.} \end{aligned}$$

On the other hand,

$$\begin{aligned} \zeta(p_s, p_b, p_w) = 0 &\Rightarrow \theta(p_s, p_b, p_w) > 0, && \text{by the above;} \\ &\Rightarrow \theta(p_s, p_b, p_s) > 0, && \text{by Theorem 3.5, since } 0 < p_w \leqslant p_s; \\ &\Rightarrow \zeta(p_s, p_b, p_s) = 0, && \text{by Theorem 3.6.} \quad \blacksquare \end{aligned}$$

Notice that in Eq. (3.26) the partition function of a model of a positive and directed animals interacting with the diagonal and superdiagonal and with contact and cycle activities, is present. The existence of a limiting free energy

$$\mathscr{F}_\zeta(x, y, z, z_s) = \lim_{v \to \infty} \frac{1}{v} \log \sum_{c, k, w, w_s} a_v(c, k, w, w_s)\, x^c y^k z^w z_s^{w_s} \tag{3.30}$$

follows from the concatenation construction in Section 2, which can be suitably adjusted to the problem here. Comparison of this with Eq. (3.26) gives Theorem 3.9 below. Moreover, it is also the case that $\mathscr{F}_\zeta(x, y, z, z_s)$ is independent of both z and z_s if both $z \leqslant 1$ and $z_s \leqslant 1$; the proof of this fact follows the same arguments leading to Theorem 2.3. That is, $\mathscr{F}_\zeta(x, y, z, z_s) = \mathscr{F}_\zeta(x, y, 1, 1)$ for all $z \leqslant 1$ and $z_s \leqslant 1$.

Theorem 3.9. $\zeta(p_s, p_b, p_w)$ and $\mathscr{F}_\zeta(x, y, z, z_s)$ are related as follows:

$$\zeta(p_s, p_b, p_w) = -\log[p_s p_b(1-p_s p_b)] - \mathscr{F}_\zeta(x, y, z, z_s),$$

where x, y, z and z_s are defined in Eqs. (3.15) and (3.25). Moreover, $\mathscr{F}_\zeta(x, y, z, z_s)$ is independent of z and z_s if both z and z_s are less than or equal to 1. ▌

The same arguments now apply to x and y as in Eqs. (3.18) and (3.19), and the discussion following it. We obtain a phase boundary separating an expanded and a collapsed phase (when xy is large) in this model. Applying Corollary 3.7 to this model shows that $\zeta(p_s, p_b, p_w) = 0$, for any value of $p_w > 0$, if (p_s, p_b) is close enough to $(1, 1)$. From Theorem 3.9 we then have

$$\mathscr{F}_\zeta(x, y, z, z_s) = -\log[p_s p_b(1-p_s p_b)], \tag{3.31}$$

and this independent of z and z_s. On the other hand, if (p_s, p_b) approach $(0, 0)$, then Theorem 3.8 implies that $\zeta(p_s, p_b, p_w) > 0$, with the result that $\mathscr{F}_\zeta(x, y, z, z_s) \neq -\log[p_s p_b(1-p_s p_b)]$ in that case. Since $xy \to 0$ as (p_s, p_b) approach $(0, 0)$, there is a non-analyticity in the limiting free energy in either x (with y fixed), or in y (with x fixed), and the directed percolation process thus underlies the collapse transitions driven by x and by y (or by both).

The original model, defined in Section 2, is recovered if $z_s = 1$, and this corresponds to the case that $p_w = p_s$ in Eq. (3.25). In this circumstance it is still the case that $z > 1$, so that a collapse phase in the model is detected when the diagonal is attractive.

As a last example, consider the case that $p_b = 1$ and $p_w = p_s$. Then by the above arguments there is a non-analyticity in $\mathscr{F}(x, y, z)$ at a critical value of p_s; say at $p_s = p_c$. Then $x = 1$, $y = 1/(1-p_s)$ and $z = 1/(1-p_s)$, and so $y \to \infty$ and $z \to \infty$ as $p_s \to 1^-$. If $p_s > p_c$, then $\mathscr{F}(1, y, z) = \log[y^2/(y-1)]$ along the locus described by $x = 1$ and $y = z = 1/(1-p_s)$, and if $p_s < p_c$, then $\mathscr{F}(1, y, z) < \log[y^2/(y-1)]$, and the transition takes place when $y = z = 1/(1-p_c) > 1$. In other words, the collapse transition takes place at a value of z which corresponds to an attractive main diagonal, but it is unclear whether the collapse is from the adsorbed phase identified in Section 2, or from the desorbed phase. Notice that the transition in this case is driven by increasing only y (and z), and is thus driven by the contact-activity, and that both $y \to \infty$ and $z \to \infty$ so that there is a collapsed phase for arbitrary large values of z, as we claimed.

On the other hand, take $p_s = p_w = 1$. Then it follows that $y = 1$ and $x = (2-p_b)/(1-p_b)$ while $z = 1/[p_b(1-p_b)]$. Taking p_b close to 1 will again give percolation at a critical value of p_b, say p_d. If $p_b > p_d$, then $\mathscr{F}(x, 1, z) = \log((x+1)^2/x) = \log z$ along the locus of points given by $y = 1$ and $x = (2-p_b)/(1-p_b)$ and $z = 1/[p_b(1-p_b)]$. If $p_b < p_d$, then $\mathscr{F}(x, 1, z) < \log((x+1)^2/x) = \log z$ along this curve. Again, the collapse transition takes place at a value of z greater than 1, and notice that in this case the transition is driven by increasing only x (and z), with $y = 1$. Thus, this transition is driven by the cycle-activity.

4. COMBINATORIAL APPROACHES

In this section we discuss several special cases in which one can find expressions for the generating function (of models of directed animals) which is equivalent to the grand canonical partition function of the model. Our motivation is to find an exact solution to the above model, or of a simplified version of it, and also to find critical exponents associated with the various phase transitions it undergoes. In the case of adsorbing directed animals (*i.e.* ignoring the cycle and contact fugacities) we are able to do so, but the problem seems intractible for collapsing directed animals. Instead, we shall examine a model of column convex directed animals, and even in this simplified model our results are incomplete. Equations (3.17) and Theorems 3.3 and 3.9 indicate that collapsing directed animals are related to percolation probability in a model of site-bond directed percolation, and this model has not been solved exactly, except in more limited models. For example, it is known that the (well studied, but still unsolved) directed bond percolation probability can be derived from the cycle generating function of directed site animals.(7)

Directed animals can be enumerated using a number of approaches: Dhar's method(17, 18) maps a collapsing animals into an interacting lattice gas, which may be solved using classical approaches in some cases. More constructive approaches relies on the Viennot notion of heaps.(2, 8, 56) These techniques can be used to enumerate directed animals in a wedge interacting with a wall, but it is not clear that they can be extended to animals with cycle or contact activities.

4.1. Enumerating Adsorbing Directed Animals

Consider the model of directed animals defined in Section 2, and put $x = y = 1$. Then only the activity z, conjugate to the number of visits in the main diagonal, remains as a parameter. This model exhibits an adsorption transition, as demonstrated in Theorem 2.3, and moreover, the critical adsorption activity is given by $z_c(1, 1) > 1$, proven in Theorem 2.5. In this subsection we find an explicit expression for the limiting free energy, and locate the critical point. We use the heap-method to do this.

Let $Z_v(z)$ be the partition function of this model, and define the generating function $G(q, z) = \sum_v Z_v(z)\, q^v$, where q is an activity conjugate to the number of vertices in the directed animal. The radius of convergence of $G(q, z)$ is $q_c(z)$, and is related to the limiting free energy by $\mathscr{F}(z) = -\log q_c(z)$. The heap method maps directed site animals into a heap-monoid of dimers (dumb-bells). In particular, there is bijection between directed animals above the main diagonal and half-pyramids of dimers. The monoid structure of the half-pyramid of dimers admits a canonical factorisation which gives the following recursive factorization for the generating function of directed animals:

$$G(q, z) = qz + qzG(q, 1) + qzG(q, 1)\, G(q, z), \tag{4.1}$$

for more details, see Bousquet-Mélou and Rechnitzer,(8) see also Rechnitzer.(54) One can immediately solve for $G(q, 1)$:

$$G(q, 1) = (1 - q - \sqrt{(1+q)(1-3q)})/2q. \tag{4.2}$$

Thus

$$G(q, z) = \frac{z(1 + (1-z)\, q - zq^2 - (1+zq)\sqrt{(1+q)(1-3q)}\,)}{2(1-(1-q)z + z^2q^2)}. \tag{4.3}$$

The singularities of physical interest can be readily found to be $q = 1/3$ and $q = (\sqrt{4z-3} - 1)/2z$). These are plotted in Fig. 7.

The curve $q_c(z)$ is given by

$$q_c(z) = \begin{cases} 1/3, & \text{if} \quad z \leqslant 3; \\ (\sqrt{4z-3} - 1)/2z, & \text{if} \quad z > 3. \end{cases} \tag{4.4}$$

The limiting free energy is simply $\mathscr{F}(1, 1, z) = -\log q_c(z)$, and the critical adsorption activity is $z_c = 3$. This point has tricritical scaling; if $z = 3$, then the generating function diverges as $G(q, 3) \sim (q_c(3) - q)^{-1/2}$ so that the exponent $\gamma_t = 1/2$ (Brak *et al.*(12) and Janse van Rensburg(41)). The cross-over exponent ϕ associated with this transition can be read from Eq. (4.4)

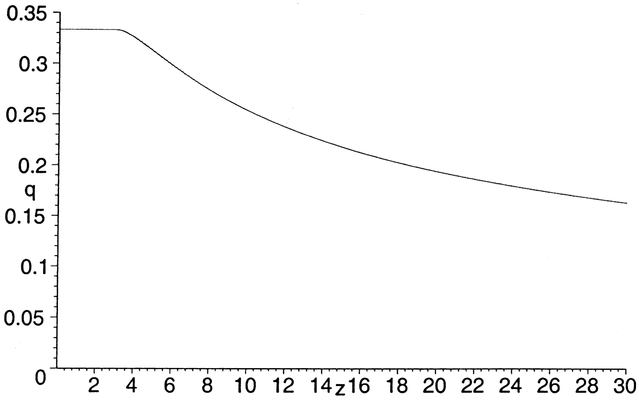

Fig. 7. The critical curve $q_c(z)$ of adsorbing directed animals.

by computing the shift exponent of $q_c(z)$ for $z > 3$ (in the adsorbed phase):

$$q_c(3) - q_c(z) = \frac{3(z-3)^2}{2z(2z+3(\sqrt{4z-3}-1))}, \tag{4.5}$$

which scales like $(z-3)^2$ as $z \to 3^+$. Thus the shift exponent is $\psi = 1/\phi = 2$. The remaining tricritical exponent can be computed from the relation $\gamma_u = \gamma_t/\phi = 1$. If $q = 1/3$ and $z < 3$ then $G(1/3, z) = 2z/(3-z)$ and this also shows that $\gamma_u = 1$, confirming tricritical scaling around the adsorption point. A further exponent along the critical curve in the adsorbed phase is defined by $G(q, z) \sim (q_c(z) - q)^{-\gamma_+}$ can be found: $\gamma_+ = 1$. Along the critical curve corresponding to the desorbed animals, there is square-root singularity in the generating function, and consequently, $\gamma_- = -1/2$.

All the values obtained here for z_c and $q_c(z)$ is consistent with the results in Section 2. The exponents are the same as those computed for adsorbing staircase walks,(41, 43) a not too suprising fact, since directed animals, Dyck paths and Motzkin words are related. By setting the contact activity to zero (*i.e.* $y = 0$) we forbid all contacts (and also all cycles), and it is not hard to see that the only remaining directed animal configurations are directed walks (above the wall). Such walks are closely related to Dyck paths and can be enumerated in a number of ways — here we shall use the Temperley method(5) to enumerate them according to the number of sites (q), the number of visits (z), and the distance between the topmost site and the wall (s). Following the methods described by Bousquet-Mélou we find that the generating function $P(s)$ satisfies:

$$P(s) = q + q\left(s + \frac{1}{s}\right)P(s) - \frac{q}{s}P(0) + q(z-1)\left(\frac{\partial P}{\partial s}\right)\Bigg|_{s=0} \tag{4.6}$$

This equation can then be solved using the kernel method[(26)] to give:

$$P(1)=\frac{q((1-z-qz)+(1-z+qz)\sqrt{(1+2q/(1-2q))}}{2(1-z+z^2q^2)}. \tag{4.7}$$

And so the critical curve is given by:

$$q_c(z)=\begin{cases}1/2, & \text{if} \quad z\leqslant 2;\\ (\sqrt{z-1})/z,\text{if} & z>2.\end{cases} \tag{4.8}$$

This critical curve is identical to that found for Dyck words and also for adsorbing staircase polygons.[(41)] The crossover exponent is still $1/2$, but the other singularities in the generating function have now changed. It can be shown that $P(1)|_{z=2}\sim(q_c(2)-q)^{-1}$ and so $\gamma_t=1$. Since the generating function is infinite on the critical curve, $q=1/2$ and $z<2$, we are not able to compute the exponent γ_u explicitly. Similarly we can show that $\gamma_-=1/2$ and $\gamma_+=1$.

4.2. Collapsing Directed Column-Convex Animals

Ideally we would be able to find the generating function of directed animals in a wedge enumerated according to the number of sites, visits, contacts and cycles (and then extract the asymptotic behaviour of the generating function). Unfortunately this seems to be a very difficult problem and we have only been able to solve simpler models. Even in this case we have not been able to solve a (non-trivial) model of animals in a wedge enumerated by sites, visits, cycles and contacts. The "closest" models we can consider are models of convex directed animals enumerated with respect to sites, cycles and contacts. The introduction of a hard wall into this model (with or without an adsorption interaction) seems to render the problem of finding the generating function far more difficult and we leave this open.

A directed animal is a directed column-convex animal, if each column (i.e., the intersection with a line $x=n$) is connected. These animals are well understood[(6, 53)] and have been enumerated according to their area and perimeter. The area generating function of this model is a simple rational function (since the heights of the columns are "independant"); adding a contact activity y introduces correlations between the heights of the columns and the generating function becomes a q-series (as we shall see below). We also introduce the activity s conjugate to the height of the last column in the animal; this variable will be useful in solving for the generating function.

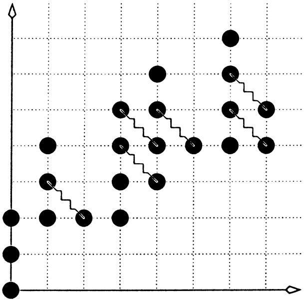

Fig. 8. An example of a directed column-convex animal.

Lemma 4.1. The generating function of directed column-convex animals, $P(s; q, y)$, satisfies the following functional equation:

$$P(s) = a(s) + b(s)\, P(1) + c(s)\, P(sqy)$$

where

$$a(s) = \frac{sq}{1 - sq}$$

$$b(s) = \frac{sq}{(1 - sq)(1 - sqy)} = \frac{sq}{(1 - sq)(1 - sQ)}$$

$$c(s) = \frac{s^2 q^2 (1 - y)}{(1 - sq)^2\, (1 - sqy)} = \frac{s^2 q (q - Q)}{(1 - sq)^2\, (1 - sQ)}.$$

where $Q = qy$.

Proof. This equation can be derived using the methods described in Bousquet-Mélou.(5) ▮

Theorem 4.2. The generating function $P(1; q, y)$ is given by

$$P(1) = \frac{A(1)}{1 - B(1)}$$

with

$$A(s) = \sum_{n \geqslant 0} c(s)\, c(sQ) \cdots c(sQ^{n-1})\, a(sQ^n),$$

$$B(s) = \sum_{n \geqslant 0} c(s)\, c(sQ) \cdots c(sQ^{n-1})\, b(sQ^n).$$

and more specifically

$$A(1) = \sum_{n \geqslant 0} \frac{q^{n+1}(q-Q)^n Q^{n^2}}{(q;Q)_n (q;Q)_{n+1} (Q;Q)_n}$$

$$B(1) = \sum_{n \geqslant 0} \frac{q^{n+1}(q-Q)^n Q^{n^2}}{(q;Q)_n (q;Q)_{n+1} (Q;Q)_{n+1}}$$

where $(z;Q)_n = (1-z)(1-zQ)\cdots(1-zQ^{n-1})$ is a q-deformation of the factorial.

Proof. Solve the equation using the iteration method described in Bousquet-Mélou.$^{(5)}$ ▮

In order to find the phase diagram of this model, we must attempt to determine the scaling behaviour of the generating function given by:

$$P(1) = \frac{\displaystyle\sum_{n \geqslant 0} \frac{q^{n+1}(q-Q)^n Q^{n^2}}{(q;Q)_n (q;Q)_{n+1} (Q;Q)_n}}{\displaystyle 1 - \sum_{n \geqslant 0} \frac{q^{n+1}(q-Q)^n Q^{n^2}}{(q;Q)_n (q;Q)_{n+1} (Q;Q)_{n+1}}}. \tag{4.9}$$

We have not been able to completely determine this. Instead, we examined $P(1)$ for $Q \leqslant q < 1$ and at $Q = 1$. Singularities in $P(1)$ are determined either by singularities in the numerator or denominator in Eq. (4.9), or there is a simple pole when the denominator is equal to zero. At some special values of Q the generating function simplifies considerably, we shall consider these points first. We shall then show that for $0 \leqslant Q < q \leqslant 1/2$ that the asymptotics of the generating function are dominated by a simple pole due to a zero demoninator in Eq. (4.9).

If $Q = 0$ then all interactions are forbidden and the only allowed animals are directed walks, because the outdegree of any vertex in the animal is either 0 or 1, and the animal is unable to branch. The generating function reduces to:

$$P(1)|_{Q=0} = \frac{q}{1-2q} \tag{4.10}$$

and so the generating function has a simple pole at $q = 1/2$. Notice also that $q_c(Q)$ is monotonic decreasing and so $q_c(Q) \leqslant 1/2$.

If $Q=q$ (or $y=1$) the generating function reduces to the area generating function of directed column-convex animals:

$$P(1)=\frac{q(1-q)}{1-3q+q^2}, \tag{4.11}$$

and the generating function has a simple pole at $q=\frac{3-\sqrt{5}}{2}\approx 0.381966\ldots$ (Klarner[(46)]). This seems to indicate that a line of simple poles exists between the pole at $y=0$ and $q=1/2$, and this pole. To see this, assume that $q<1$ and let $Q<q$. It can be shown that the denominator in Eq. (4.9) is uniformly convergent for both $Q\leqslant r_0$ and $q\leqslant r_0$ where $r_0<1$ is arbitrary. Thus, the denominator in Eq. (4.9) is a continuous function for $(q,Q)\in[0,1)\times[0,1)$. If $0\leqslant Q\leqslant q\leqslant 1/2$ then each term in $B(1)$ is a series with positive terms and is convergent. $B(1)$ can be bound from below by its first term which is $b(1)=q/(1-q)(1-Q)$. This lower bound is equal to 1 if $q=(1-Q)/(2-Q)$, thus the denominator of $P(1)$ is zero for some value of $q\leqslant(1-Q)/(2-Q)\leqslant 1/2$. This gives a simple pole in $P(1)$.

If $Q=1$ then the above expression is singular, however one can obtain a non-singular expression by returning to the original functional equation and solving it using the kernel method:[(26)]

$$P(s)=\frac{sq}{1-sq}+\frac{sq}{(1-s)(1-sq)}P(1)+\frac{s^2q(q-1)}{(1-s)(1-sq)^2}P(s) \tag{4.12}$$

We solve for $P(1)$ by moving all the $P(s)$ terms to the left hand-side of the equation:

$$P(s)\left(1-\frac{s^2q(q-1)}{(1-s)(1-sq)^2}\right)=\frac{sq}{1-sq}+\frac{sq}{(1-s)(1-sq)}P(1). \tag{4.13}$$

We then choose a value of $s=s_0$ such that the coefficient of $P(s)$ in this equation is equal to 0. There is only one value of s such that $(1-s)(1-sq)^2-s^2q(q-1)=0$, whose coefficients (with respect to q and Q) are real and we define this to be s_0. Setting $s=s_0$ in this equation gives:

$$P(1)=s_0-1. \tag{4.14}$$

If $q=5/32$ and $Q=1$ then this function behaves as $P(1)\sim(y-32/5)^{-\gamma_u}$ with $\gamma_u=-1/2$. The remaining exponents seem difficult to extract, even if the original functional recursion is examined instead.

In the case that $Q>q$ the series $B(1)$ is alternating, but still absolutely convergent, and even uniformly convergent for $Q\leqslant r_0<1$. However, we were unable to show that $B(1) = 1$ for some value of $q < 1/2$ and fixed $q < Q < 1$. If one puts $Q = yq$ in the above, then it can be shown, using techniques similar to those in Section 2.1, that the limiting free energy of directed column-convex animals exists, and it has limiting free energy defined the radius of convergence of $P(1)$. Moreover, it is a convex function of $\log y$, and there is a critical point y_c such that the radius of convergence is given by

$$q_c(y)\begin{cases} = 1/y, & \text{if} \quad y > y_c; \\ < 1/y, & \text{if} \quad y < y_c. \end{cases} \tag{4.15}$$

One might guess that $y_c = 32/5$ from the above, but even this is not known. The above also show that the limiting free energy is non-analytic, and that a collapse transition takes place at $y = y_c \leqslant 32/5$.

5. CONCLUSIONS

Although classic directed animals without additional statistics are well understood, it appears that the addition of new properties such as contacts, or cycles, makes the problem very difficult to deal with. The introduction of a geometric constraint, such as a hard wall, is another compounding factor. In its own right, this problem is therefore interesting from a combinatorial point of view, and we have only managed to solve it in the very special case with the presense of an adsorption generating variable in Section 4.1. Our attempts to find a full solution for the model which includes generating variables for contacts, cycles and contacts proved unsuccessful, and we considered the more limited model of a column convex animal. Even in that case, we are only able to find exact expressions only for specific values of the activities, and so our results in Section 4.2 remain incomplete.

The full model can be studied perhaps more fruitfully by using analytic techniques, which we did in Sections 2 and 3. In Section 2 we showed explicitly the presence of a critical surface $z_c(x, y)$ of adsorption transitions in the phase diagram. If the collapse activities are schwitched off ($x = y = 1$), then the location of the critical adsorption point is given by Eq. (4.8), $z_c = 3$. The crossover exponents associated with the adsorption is $\phi = 1/2$; which is the value usually associated with adsorption transitions in models of adsorbing random systems. This value for ϕ is of course obtained if expanded (not collapsed) directed animals undergo adsorption, and should retain the same value along the entire critical adsorption curve separating desorbed expanded directed animals from adsorbed (and expanded) directed animals. While we are uncertain about the shape of the critical adsorption curve, we were able to prove that $z_c(x, y) > 1$ (Theorem 2.5).

The collapse transition in this model can be examined by using a model of directed site-bond percolation. In Section 3.1 we examine this transition for desorbed directed animals, and found a critical surface of collapsed transitions (in the xyz-parameter space). By Theorem 2.6 this surface is a constant function of z, and together with the adsorption critical surface bound the phase of expanded desorbed directed animals.

While the results in Section 3.1 indicates a collapsed transition from desorbed directed animals, it seems intuitive that a collapse transition should also be possible from adsorbed directed animals. This question is addressed in Section 3.2. By examining directed percolation in a wedge, the presence of a collapsed phase of directed animals, for arbitrarily large values of the adsorption activity z, is shown. It is unclear from our results whether this is a phase of adsorbed-collapsed, or of desorbed-collapsed, directed animals. This can only be done if more about the density function for the full model is known: in particular, is there a density of visits for arbitrarily large x or y, and z fixed and large?

ACKNOWLEDGMENTS

E.J.J.v.R. is supported by a grant from NSERC Canada.

REFERENCES

1. M. T. Batchelor and C. M. Yung, *Phys. Rev. Lett.* **74**:2026 (1995).
2. J. Bétréma and J.-G. Penaud, *Theor. Comp. Sci.* **117**:67 (1993).
3. M. Bousquet-Mélou, *J. Phys. A: Math. Gen.* **25**:1925 (1992).
4. M. Bousquet-Mélou, *Disc. Appl. Math.* **48**:21 (1994).
5. M. Bousquet-Mélou, *Disc. Math.* **154**:1 (1996).
6. M. Bousquet-Mélou, *Rapport d'habilitation*, LaBRI Université Bordeaux 1 (1996a).
7. M. Bousquet-Mélou, *Disc. Math.* **180**:73 (1998).
8. M. Bousquet-Mélou and A. Rechnitzer, submitted to *Disc. Math.* (2000).
9. R. Brak, J. M. Essam, and A. L. Owczarek, *J. Stat. Phys.* **93**:155 (1998).
10. R. Brak, A. J. Guttmann, and S. G. Whittington, *J. Math. Chem.* **8**:255 (1991).
11. R. Brak, A. J. Guttmann, and S. G. Whittington, *J. Phys. A: Math. Gen.* **25**:2437 (1992).
12. R. Brak, A. L. Owczarek, and T. Prellberg, *J. Stat. Phys.* **76**:1101 (1994).
13. J. L. Cardy, *Phase Transitions and Critical Phenomena*, Vol. 11, C. Domb and J. L. Lebowitz, eds. (Academic Press, London, 1987), p. 55.
14. M. C. T. P. Carvalho and V. Privman, *J. Phys. A: Math. Gen. 21* L1033 (1988).
15. K. De'Bell and T. Lookman, *Rev. Mod. Phys.* **65**:87 (1993).
16. B. Derrida and H. J. Hermann, *J. Physique* **46**:1365 (1983).
17. D. Dhar, *Phys. Rev. Lett.* **49**:959 (1982).
18. D. Dhar, *Phys. Rev. Lett.* **51**:853 (1983).
19. C. Domb, *J. Phys. A: Math. Gen.* **9**:L141 (1976).

20. B. Duplantier and H. Saleur, *Phys. Rev. Lett.* **57**:3179 (1986).
21. E. Eisenreigler, *J. Chem. Phys.* **79**:1052 (1983).
22. E. Eisenreigler, *J. Chem. Phys.* **82**:1032 (1985).
23. E. Eisenreigler, K. Kremer, and K. Binder, *J. Chem. Phys.* **77**:6296 (1982).
24. R. S. Ellis, *Entropy, Large Deviations and Statistical Mechanics* (Springer, New York, 1985).
25. M. E. Fisher, *J. Stat. Phys.* **34**:667 (1984).
26. P. Flajolet and R. Sedgewick, *Analytic Combinatorics: Functional Equations, Rational and Algebraic Functions*, INRIA Rapport de recherche No. 4103 (2001).
27. S. Flesia and D. S. Gaunt, *J. Phys. A: Math. Gen.* **25**:2127 (1992).
28. S. Flesia, D. S. Gaunt, C. Soteros, and S. G. Whittington, *J. Phys. A: Math. Gen.* **25**:3515 (1992).
29. S. Flesia, D. S. Gaunt, C. Soteros, and S. G. Whittington, *J. Phys. A: Math. Gen.* **26**:L993 (1993).
30. S. Flesia, D. S. Gaunt, C. Soteros, and S. G. Whittington, *J. Phys. A: Math. Gen.* **27**:5831 (1994).
31. D. S. Gaunt and S. Flesia, *Physica A* **168**:602 (1990).
32. D. S. Gaunt and S. Flesia, *J. Phys. A: Math. Gen.* **24**:3655 (1991).
33. G. Grimmett, *Percolation* (Springer, Berlin, 1989).
34. J. M. Hammersley, *Math. Proc. Camb. Phil. Soc.* **56**:642 (1957).
35. J. M. Hammersley, *Math. Proc. Camb. Phil. Soc.* **57**:516 (1961).
36. J. M. Hammersley, G. M. Torrie, and S. G. Whittington, *J. Phys. A: Math. Gen.* **15**:539 (1982).
37. J. M. Hammersley and S. G. Whittington, *J. Phys. A: Math. Gen.* **18**:101 (1985).
38. F. Harary, *MagyarTud. Akad. Mat. Kutató Intezetenek Kozlemanyei* **5**:63 (1960).
39. Hille, *Functional Analysis and Semi-Groups. AMS Colloquim Publications*, Vol. 31, American Mathematical Society (Providence, Rhode Island, 1948).
40. E. J. Janse van Rensburg, *J. Phys. A: Math. Gen.* **31**:8295 (1998).
41. E. J. Janse van Rensburg, *Ann. Comb.* **3**:451 (1999).
42. E. J. Janse van Rensburg, *J. Phys. A: Math. Gen.* **33**:3653 (2000).
43. E. J. Janse van Rensburg, *The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles*, Oxford Series in Mathematics and Its Applications, Vol. 18 (Oxford University Press, 2000a).
44. E. J. Janse van Rensburg, *J. Phys. A: Math. Gen.* **33**:3653 (2000b).
45. E. J. Janse van Rensburg and S. You, *J. Phys. A: Math. Gen.* **31**:8635 (1998).
46. D. Klarner, *Fibonacci Quart.* **3**:9 (1965).
47. D. Klarner, *Can. J. Math.* **19**:851 (1967).
48. T. C. Lubensky and J. Isaacson, *Phys. Rev.* **20**:2130 (1979).
49. G. Pólya, *J. Comb. Theo.* **6**:102 (1969).
50. T. Prellberg and R. Brak, *J. Stat. Phys.* **78**:701 (1995).
51. V. Privman, G. Forgacs, and H. L. Frisch, (1988). *Phys. Rev. B* **37**:9897.
52. V. Privman and N. M. Švrakić, *J. Stat. Phys.* **51**:1091 (1988).
53. V. Privman and N. M. Švrakić, *Phys. Rev. Lett.* **60**:1107 (1988a).
54. A. Rechnitzer, *Some Problems in the Counting of Lattice Animals, Polyominoes, Polygons and Walks*, Ph.D. thesis (University of Melbourne, 2000).
55. D. Stauffer, *Phys. Rep.* **54**:1 (1979).
56. G. X. Viennot, *Combinatoire énumérative*, G. Labelle and P. Leroux, eds., *Lect. Notes in Math.*, Vol. 1234 (1985).
57. S. You and E. J. Janse van Rensburg, *J. Phys. A: Math. Gen.* **33**:1171 (2000).